5.5 Multiple moving average

In the case of applying two times,

$$g_i = \sum_m w_m f_{i-m}$$

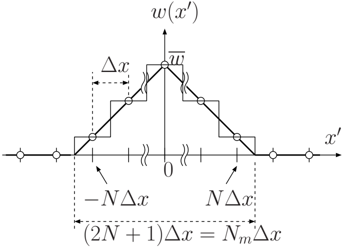
$$h_i = \sum_{m'} w_{m'} g_{i-m'}$$

$$(g_{i-m'} = \sum_m w_m f_{i-m'-m})$$

$$= \sum_{m'=-N}^{N} \sum_{m=-N}^{N} w_{m'} w_m f_{i-m'-m}$$

When $N_m = 3(N = 1), w_m = w'_m = 1/3$

$$g_i = \frac{f_{i-2} + 2f_{i-1} + 3f_{i-1} + 2f_{i+1} + f_{i+2}}{9}$$



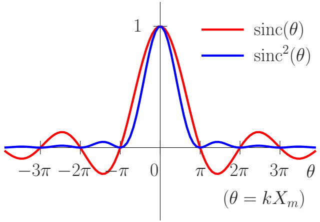
Applying multiple moving average is equivalent to moving average with which central weight is larger than neighboring points.

Spectral Gain of Multiple Moving Average

In the case of two times,

$$g(x) = \int_{-\infty}^{\infty} w(x')f(x-x')\,dx' \quad \rightarrow \quad G(k) = W(k)F(k)$$

$$h(x) = \int_{-\infty}^{\infty} w(x')g(x-x')\,dx' \quad \rightarrow \quad H(k) = W(k)G(k) = \underline{W^2(k)}F(k)$$

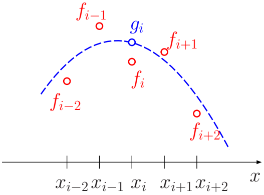


- Further reduction of higher frequency components is applied.
- No spurious resolution.

5.6 Higher order Moving average (Savitzky-Golay filter)

For each i,

- Represent the smoothed function $g_i(x_j)$ by a power series expansion.
  $$(j \in \{i-N, \cdots, i+N\})$$
  (In the case of second order expansion,)
  $$g_i(x_j) = a_i(x_j - x_i)^2 + b_i(x_j - x_i) + c_i$$
  $$= a_i \Delta x_{j,i}^2 + b_i \Delta x_{j,i} + c_i$$



- Using by the least square method, determine the parameters a_i, b_i, and c_i which are the parameters of the fitting function $g_i(x_j)$.
- The moving average at the point i corresponds to the value of the fitting function at the $x_j = x_i$; i.e $g_i(x_i) = c_i$.

Least square fitting to the parabolic function

Fitting function

$$g_j = a\Delta x_j^2 + b\Delta x_j + c \qquad \text{(omit } i\text{)}$$

Minimize Average of square residual, E:

$$E(a,b,c) = \frac{1}{N_m}\sum_{j=i-N}^{i+N}(g_j - f_j)^2 \equiv \overline{(g_j - f_j)^2}$$

$$\text{minimize } E(a,b,c) \quad \Longleftrightarrow \quad \frac{\partial E}{\partial \xi} = 0 \quad (\xi \in \{a,b,c\})$$

$$\left(\begin{array}{c} \frac{\partial E}{\partial \xi} = \frac{\partial}{\partial \xi}\overline{(g_j(\xi) - f_j)^2} = 2\overline{(g_j(\xi) - f_j)\frac{\partial g_j(\xi)}{\partial \xi}} \\ \frac{\partial g_j}{\partial a} = \Delta x_j^2,\ \frac{\partial g_j}{\partial b} = \Delta x_j,\ \frac{\partial g_j}{\partial c} = 1 \end{array} \right)$$

$$\begin{pmatrix} \overline{\Delta x_j^4} & \overline{\Delta x_j^3} & \overline{\Delta x_j^2} \\ \overline{\Delta x_j^3} & \overline{\Delta x_j^2} & \overline{\Delta x_j} \\ \overline{\Delta x_j^2} & \overline{\Delta x_j} & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \overline{f_j \Delta x_j^2} \\ \overline{f_j \Delta x_j} \\ \overline{f_j} \end{pmatrix}$$

Least square fitting to the power series

Fitting function

$$\widetilde{f}(x;\boldsymbol{a}) = \sum_{l=0}^{N_l} a_l x^l,$$

$$\boldsymbol{a} = (a_0, a_1, \cdots, a_{N_l})$$

Sampling points (known)

$$(x_i, f_i),\ i \in \{1, \cdots, N\}$$

Minimize Average of square residual, E

$$E(\boldsymbol{a}) = \frac{1}{N_i}\sum_{i=1}^{N_i}(\widetilde{f}(x_i;\boldsymbol{a}) - f_i)^2$$

$$\equiv \overline{(\widetilde{f}(x_i;\boldsymbol{a}) - f_i)^2}$$

minimize $E(\boldsymbol{a}) \quad \Leftrightarrow \quad \frac{\partial E}{\partial a_l} = 0$

$$\left(\begin{aligned} \frac{\partial E}{\partial a_l} &= \frac{\partial}{\partial a_l}\overline{(\widetilde{f}_i - f_i)^2} = 2\overline{(\widetilde{f}_i - f_i)\frac{\partial \widetilde{f}_i}{\partial a_l}} \\ &= 2\left(\overline{\widetilde{f}_i x_i^l} - \overline{f_i x_i^l}\right) = 0 \quad \left(\because \frac{\partial \widetilde{f}_i}{\partial a_l} = x^l\right) \end{aligned} \right)$$

$$\therefore \qquad \sum_{l=0}^{N_l} \overline{x_i^{l+m}} a_l = \overline{f_i x_i^m}$$

$$\begin{pmatrix} \overline{x^0} & \overline{x^1} & \cdots & \overline{x^{N_l}} \\ \overline{x^1} & \overline{x^2} & \cdots & \overline{x^{N_{l+1}}} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{x^{N_l}} & \overline{x^{N_{l+1}}} & \cdots & \overline{x^{N_{2l}}} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{N_l} \end{pmatrix} = \begin{pmatrix} \overline{f_i x_i^0} \\ \overline{f_i x_i^1} \\ \vdots \\ \overline{f_i x_i^{N_l}} \end{pmatrix}$$

The parameter of the least square fitting to power series function (non-linear function) can be obtained by solving a set of linear equations.

Moving average by parabolic fitting

$$\begin{pmatrix} \overline{\Delta x_j^4} & \overline{\Delta x_j^3} & \overline{\Delta x_j^2} \\ \overline{\Delta x_j^3} & \overline{\Delta x_j^2} & \overline{\Delta x_j} \\ \overline{\Delta x_j^2} & \overline{\Delta x_j} & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \overline{f_j \Delta x_j^2} \\ \overline{f_j \Delta x_j} \\ \overline{f_j} \end{pmatrix}$$

In the case of $N_m = 5(N = 2)$

$$\left(\begin{array}{l} \overline{\Delta x_j} = 0,\ \overline{\Delta x_j^3} = 0, \\ \overline{\Delta x_j^2}/\Delta^2 = \frac{(-2)^2+(-1)^2+0^2+(+1)^2+(+2)^2}{5} = \frac{2(1^2+2^2)}{5} = 2, \\ \overline{\Delta x_j^4}/\Delta^4 = \frac{2(1^4+2^4)}{5} = \frac{34}{5} \end{array} \right)$$

$$c = \begin{vmatrix} \frac{34}{5}\Delta^4 & 0 & \overline{f_j \Delta x_j^2} \\ 0 & 2\Delta^2 & \overline{f_j \Delta x_j} \\ 2\Delta^2 & 0 & \overline{f_j} \end{vmatrix} \Bigg/ \begin{vmatrix} \frac{34}{5}\Delta^4 & 0 & 2\Delta^2 \\ 0 & 2\Delta^2 & 0 \\ 2\Delta^2 & 0 & 1 \end{vmatrix}$$

$$= \frac{\frac{68}{5}\Delta^6 \overline{f_j} - 4\Delta^4 \overline{f_j \Delta x_j^2}}{\frac{28}{5}\Delta^6}$$

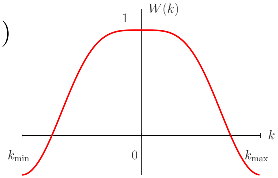
$$\text{Num.} = \frac{68[f_{-2} + f_{-1} + f_0 + f_1 + f_2]}{5}\Delta^6 - \frac{4[(-2)^2 f_{-2} + (-1)^2 f_{-1} + 0^2 f_0 + 1^2 f_1 + 2^2 f_2]\Delta^2}{5}\Delta^4$$

$$= \Delta^6 \left(\left(\frac{68}{25} - \frac{16}{5}\right)(f_{-2} + f_{+2}) + \left(\frac{68}{25} - \frac{4}{5}\right)(f_{-1} + f_{+1}) + \frac{68}{25} f_0 \right)$$

$$g_i = c = \frac{1}{35} \begin{pmatrix} -3 & 12 & 17 & 12 & -3 \end{pmatrix} \begin{pmatrix} f_{i-2} \\ f_{i-1} \\ f_i \\ f_{i+1} \\ f_{i+2} \end{pmatrix}$$

- The weight at point i is maximum.
- The weights at both ends are negative.

$W(k)$



- $W(k)$ is flat in low frequency.

5.7 Gaussian Filter

Gaussian filter:

$\equiv$ Moving average with which the weight, $w(x')$, is a Gaussian function.

$$w(x) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{x^2}{2\sigma_x^2}} \quad \overset{\mathcal{F}}{\Longrightarrow} \quad W(k) = e^{-\frac{k^2}{2\sigma_k^2}} \quad \left(\sigma_k = \frac{1}{\sigma_x}\right)$$

(proof is shown in the next page.)

- $W(k)$ is a simple decreasing function.
- $W(k) > 0 \quad \rightarrow \quad$ No spurious resolution.

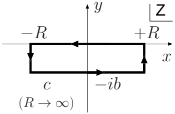
Fourier Transform of Gaussian function

$$
\begin{aligned}
W(k) &= \mathcal{F}\{w(x)\} \\
&= \frac{1}{\sqrt{2\pi}\sigma_x}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma_x^2}} e^{-ikx}\,dx \\
&\quad \left(X = \frac{x}{\sqrt{2}\sigma_x}\right) \\
&= \frac{1}{\sqrt{\pi}} e^{-\frac{\sigma_x^2}{2}k^2} \underbrace{\int_{-\infty}^{\infty} e^{-(X+i\frac{k\sigma_x}{\sqrt{2}})^2}\,dX}_{=\int e^{-X^2}dX=\sqrt{\pi} \quad (*)} \\
&= e^{-\frac{\sigma_x^2}{2}k^2} = e^{-\frac{1}{2\sigma_k^2}k^2} \qquad \left(\sigma_k = \frac{1}{\sigma_x}\right)
\end{aligned}
$$

$$
\boxed{\mathcal{F}\left\{e^{-\frac{x^2}{2\sigma_x^2}}\right\} \propto e^{-\frac{\sigma_x^2 k^2}{2}}}
$$

(*)

$$
\begin{aligned}
I &= \int_{-\infty}^{\infty} e^{-(x+ib)^2}\,dx \\
&= \int_{-\infty-ib}^{\infty-ib} e^{-z^2}\,dz
\end{aligned}
$$



(No poles)

$$
\int_c e^{-z^2}\,dz = 0 \quad \rightarrow I = \int_{-\infty}^{\infty} e^{-x^2}\,dx
$$

$$
\begin{aligned}
I^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy \\
&= \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\,d\theta\,dr \qquad (t = r^2) \\
&= -\pi\left[e^{-t}\right]_0^{\infty} = \pi \\
&\therefore I = \sqrt{\pi}
\end{aligned}
$$

5.8 Cummulative Average of muliple measurement

- N_k sets of measurement.

$$f_i^{(k)} = \widetilde{f}_i + n_i^{(k)}$$
$$(k \in \{1, \cdots, N_k\})$$

- Cummulative Ave.

$$\langle\langle y_i \rangle\rangle \equiv \frac{1}{N_k} \sum_{k=1}^{N_k} y_i^{(k)}$$

- Cummul. Ave. of f_i

$$g_i = \langle\langle f_i \rangle\rangle = \widetilde{f}_i + \langle\langle n_i \rangle\rangle$$

- Expected value : $\mathrm{E}[g_i]$

$$\mathrm{E}[g_i] = \widetilde{f}_i + \underbrace{\mathrm{E}[\langle\langle n_i \rangle\rangle]}_{=\langle\langle \mathrm{E}[n_i] \rangle\rangle = 0}$$
$$= \widetilde{f}_i$$

- Variance of g_i : $\sigma_{g_i}^2$ (※ $\neq$ Var. of f_i)

$$\sigma_{g_i}^2 = \mathrm{E}\left[(g_i - \mathrm{E}[g_i])^2\right] = \mathrm{E}\left[\left\{(\widetilde{f}_i + \langle\langle n_i \rangle\rangle) - \widetilde{f}_i\right\}^2\right]$$
$$= \mathrm{E}\left[\langle\langle n_i \rangle\rangle^2\right] = \mathrm{E}\left[\frac{1}{N_k^2} \sum_k \sum_{k'} n_i^{(k)} n_i^{(k')}\right]$$
$$= \mathrm{E}\left[\frac{1}{N_k^2} \sum_k \left(\left(n_i^{(k)}\right)^2 + \sum_{k' \neq k} n_i^{(k)} n_i^{(k')}\right)\right]$$
$$= \frac{\sum_k \overbrace{\mathrm{E}\left[\left(n_i^{(k)}\right)^2\right]}^{\sigma_n^2}}{N_k^2} + \frac{\sum_k \sum_{k' \neq k} \overbrace{\mathrm{E}\left[n_i^{(k)} n_i^{(k')}\right]}^{0}}{N_k^2}$$
$$= \frac{1}{N_k^2} \sum_{k=1}^{N_k} \sigma_n^2 = \frac{\sigma_n^2}{N_k}$$

$\sqrt{\sigma_{g_i}^2}$ is called standard error.

Comparison of Cummulative Ave. and Moving Ave.

	Original	Moving Average	Cummulative Average
Ave.	$\widetilde{f_i}$	$\left\langle \widetilde{f_i} \right\rangle_m$	$\widetilde{f_i}$
Var.*	σ_n^2	$\dfrac{\sigma_n^2}{N_m}$	$\dfrac{\sigma_n^2}{N_k}$

(
* for White Noise
N_m : Number of averaging samples
N_k : Number of times of measurement
)

- The expected value of average is distorted by moving average, but not distorted by cummulative average.
- The variances become smaller for both averaging.
- If we can obtain measurements under same condition, cummlative average is superior.

5.9 Propagation of Error

Two independent measurements, f and g ($|df| \ll |\widetilde{f}|$, $|dg| \ll |\widetilde{g}|$) :

$$f = \widetilde{f} + df, \quad \mathrm{E}[f] = \widetilde{f}, \quad \mathrm{E}[df] = 0, \quad \mathrm{E}\left[(df)^2\right] = \sigma_f^2$$
$$g = \widetilde{g} + dg, \quad \mathrm{E}[g] = \widetilde{g}, \quad \mathrm{E}[dg] = 0, \quad \mathrm{E}\left[(dg)^2\right] = \sigma_g^2$$

Consider evaluation of a new result : $h \equiv h(f,g) = \widetilde{h}(f,g) + dh(f,g)$

- Average:

$$dh = \left.\frac{\partial h}{\partial f}\right|_{(\widetilde{f},\widetilde{g})} df + \left.\frac{\partial h}{\partial g}\right|_{(\widetilde{f},\widetilde{g})} dg$$
$$= h'_f df + h'_g dg$$
$$\mathrm{E}[dh] = h'_f \mathrm{E}[df] + h'_g \mathrm{E}[dg]$$
$$= 0$$
$$\mathrm{E}[h] = \mathrm{E}\left[\widetilde{h} + dh\right] = \widetilde{h}(\widetilde{f},\widetilde{g})$$

- Variance:

$$\sigma_h^2 = \mathrm{E}\left[(dh)^2\right] = \mathrm{E}\left[\left(h'_f df + h'_g dg\right)^2\right]$$
$$= {h'_f}^2 \underbrace{\mathrm{E}\left[df^2\right]}_{\sigma_f^2} + {h'_g}^2 \underbrace{\mathrm{E}\left[dg^2\right]}_{\sigma_g^2} + 2\, h'_f h'_g \underbrace{\mathrm{E}[df \cdot dg]}_{=0}$$
$$= \left(\left.\frac{\partial h}{\partial f}\right|_{\widetilde{h}}\right)^2 \sigma_f^2 + \left(\left.\frac{\partial h}{\partial g}\right|_{\widetilde{h}}\right)^2 \sigma_g^2$$

Example of Error Propagation

$$h \equiv h(f,g)$$
$$\sigma_h^2 = \left(\frac{\partial h}{\partial f}\bigg|_{\widetilde{h}} \right)^2 \sigma_f^2 + \left(\frac{\partial h}{\partial g}\bigg|_{\widetilde{h}} \right)^2 \sigma_g^2$$

- Add. and Sub.
  - $h = f + g$
    $$\sigma_h^2 = \sigma_{f+g}^2 = \sigma_f^2 + \sigma_g^2$$
  - $h = f - g$
    $$\sigma_h^2 = \sigma_{f-g}^2 = \sigma_f^2 + \sigma_g^2$$

  Sum of each Variance.

- Mul. and Div.
  - $h = f \cdot g$
    $$\sigma_h^2 = \sigma_{f \cdot g}^2 = g^2\sigma_f^2 + f^2\sigma_g^2$$
    $$\frac{\sigma_h^2}{h^2} = \frac{\sigma_f^2}{f^2} + \frac{\sigma_g^2}{g^2}$$
  - $h = f/g$
    $$\sigma_h^2 = \sigma_{f/g}^2 = \frac{1}{g^2}\sigma_f^2 + \frac{f^2}{g^4}\sigma_g^2$$
    $$\frac{\sigma_h^2}{h^2} = \frac{\sigma_f^2}{f^2} + \frac{\sigma_g^2}{g^2}$$

  Sum of each normalized Variance.

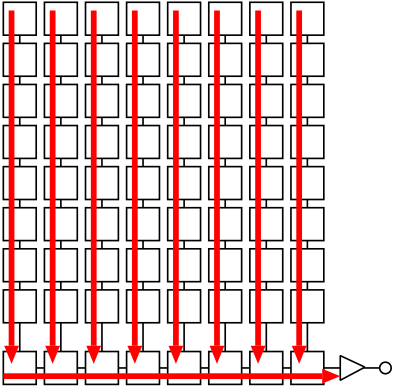
6. Image data
6.1 Image sensor

- Image Sensor
  - CCD (Charge Coupled Device)
  - CMOS (Complementary MOS(Metal Oxide Semiconductor))

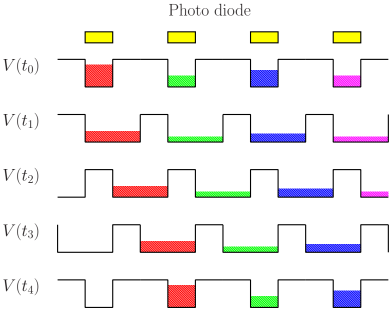
	CCD	CMOS
detection	Photo diode (Photo electron emission/excitation)	
Sensitivity	High	Low(High recently)
Amplifier, A/D converter	one for whole pixels	one for each pixel
Readout	Bucket relay (only whole pixels reading)	Addressing (possible to read one pixel)
Defect pixel	None	Existing

CCD

Circuit

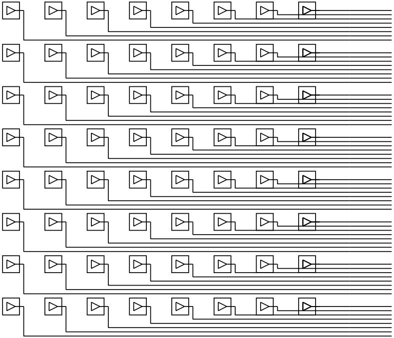


Bucket Relay



CMOS

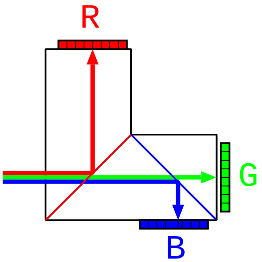
Circuit



Color Camera

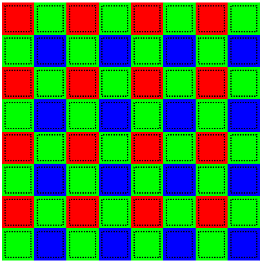
- three chips (high quality)

  Separating color by using prisms, each of which can reflect a certain color. Separated beams are detected each sensor.



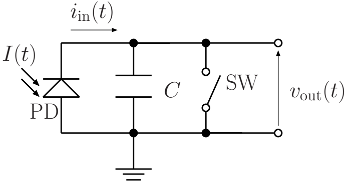
- one chip (small size)

  Color filter is placed in front of sensor.



6.2 Signal to Noise ratio and Resolution

- Charge accumuration by photo diode



- Incident Light $I(t) \propto i_{\rm in}(t)$
  - SW=ON(close) : $v_{\rm out}(t) = 0$
  - SW=OFF(open) $(t = [0, T])$

    $v_{\rm out}(T) = \frac{1}{C} \int_0^T i_{\rm in}(t)\, dt$

    If $I(t) = \overline{I}$ (const),

    $v_{\rm out}(T) = k\overline{I}T$.

    Signal is proportional to T.

- I includes fluctuation :

  $I(t) = \overline{I} + \delta I(t) \to \overline{I} \pm \sigma_{\delta I}$

  $\delta v_{\rm out}(T) = k \int_0^T \delta I(t)\, dt$

  $\frac{1}{T} \int_0^T \delta I(t)\, dt = \langle\!\langle \delta I \rangle\!\rangle$

  $\equiv$ Cummul. ave. of $\delta I(t)$

  $\sigma_{\langle\!\langle \delta I \rangle\!\rangle} \propto \sigma_{\delta I}/\sqrt{T}$ (δI is white)

  $\sigma_{v_{\rm out}}(T) = T\sigma_{\langle\!\langle \delta I \rangle\!\rangle} = k' \sigma_{\delta I} \sqrt{T}$

  Noise is proportional to $\sqrt{T}$.

Signal to Noise Ratio S/N :

$S/N \propto \sqrt{T}$

The quality of signal increases with increasing T.

- Area of Pixel A

$$i_{\text{in}}(t) \propto \int_A I(t,x,y)\,dA$$

  - $I(t,x,y) = \overline{I}$:

    $v_{\text{out}}(A) = k\overline{I}A$

    v_{out} is proportional to A.

  - $I(t,x,y) = \overline{I} + \delta I(t,x,y)$:

    $\delta v_{\text{out}} = k\int_A \delta I(t,x,y)\,dA$

    $\sigma_{\langle\langle \delta I\rangle\rangle} \propto \sigma_{\delta I}/\sqrt{A}$

    $\sigma_{v_{\text{out}}}(A) = k'\sigma_{\delta I}\sqrt{A}$

    $\sigma_{v_{\text{out}}}$ is proportional to $\sqrt{A}$.

Signal to Noise Ratio S/N :

$$S/N \propto \sqrt{A}$$

The quality of signal increases with increasing pixel size A.

- Resolution
  - Temporal resolution ⇔Exposure time
  - Spatial resolution ⇔Pixel Size

	S/N Larger is better	Resolution Smaller is better
time	$\propto \sqrt{T}$	T
space	$\propto \sqrt{A}$	$\sqrt{A}$

6.3 Discretization and Quantization

- Discretization and Quantization
  Continuous function $f(x)$
  $(x \in \mathbb{R},\ f(x) \in \mathbb{R})$
  - Discretization:
    (Digitizing Domain)
    $x_n = n\Delta x,\ n \in \mathbb{Z}$
  - Quantization:
    (Digitizing Range of f)
    $f_m = m\Delta f,\ m \in \mathbb{Z}$

- Image Data
  - Pixel : Discrete point
    $\Leftrightarrow \quad i, j \in \mathbb{Z}$
    (e.g. 640×400, 1024×768)
  - Intensity : Quantized of brightness
    $\Leftrightarrow \quad I_{i,j} \in \mathbb{Z}$
    A/D (Analog to Digital) converter

    e.g.

8bits	$(0, \cdots, 255)$
10bits	$(0, \cdots, 1023)$
12bits	$(0, \cdots, 4095)$

6.4 Correction of Intensity

γ correction

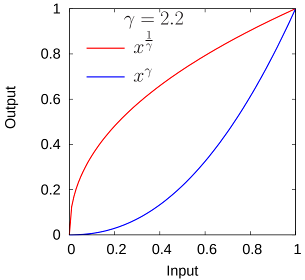
$$I_i \overset{\text{Record}}{\Longrightarrow} f_r \overset{\text{Display}}{\Longrightarrow} I_o$$

- Display device
  Gain of CRT is not linear
  $\rightarrow I_o \not\propto f_r$
  $I_o \propto f_r^{\gamma_d}$ (e.g. $\gamma_d = 2.2$)
- Recording device
  $f_r \propto I_i^{\gamma_r}$

$$\therefore I_o = I_i^{\gamma_r \cdot \gamma_d}$$

- In general, recording device has $\gamma_r = 1/\gamma_d$ so that $I_o = I_i$.
- This is not appropriate to quantitative processing.
- If we wish quantitative evaluation, apply the cancelation of the γ correction

$$\widehat{f} = f_r^{1/\gamma_r} \propto I_i.$$



6.5 Image format

- Single format

Format	Name	Color	bit	Compres.	Revers.	Multi-image
PBM	Portable Bit Map	White/Black	1	×	○	×
PGM	Portable Gray Map	Gray	8	×	○	×
PPM	Portable Pixel Map	RGB	3×8	×	○	×
GIF	Graphics Interchange Format	RGB	3×8	○	○	○
JPEG	Joint Photographic Experts Group	RGB	3×8	○	×	×
PNG	Portable Network Graphics	RGB-alpha*	4×16	○	○	×

* : alpha is a channel for transparency

- Integrated Multiple formats

Format	Name
PNM	Portable aNy Map (PBM, PGM, PPM)
TIFF	Tagged Image File Format
BMP	Microsoft windows BitMaP