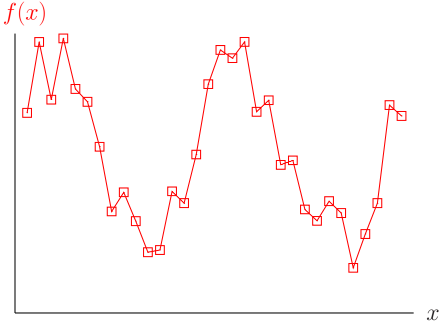
# 5. Noise Reduction using Moving average

Observed data including noise

$f(x_i) = \tilde{f}(x_i) + n(x_i)$

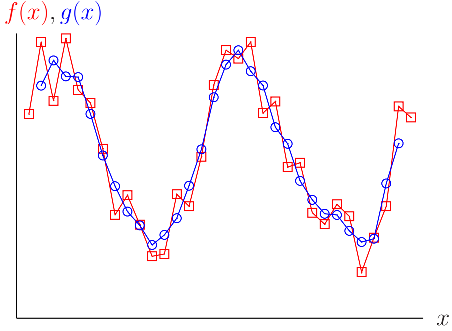
$(i \in \{1, \cdots, N_{all}\})$

| Ave. Method | Num. of Ave. |
|---|---|
| whole data | 1 |
| Ave. for $M$ groups | $M < N_{all}$ |
| Moving Average | $N'_{all} \sim N_{all}$ |



# 5.1 Moving Average

Averages are computed while replacing a part of samples.




Moving average with 3 points

$$g_2 = (f_1 + f_2 + f_3)/3$$
$$g_3 = (f_2 + f_3 + f_4)/3$$
$$g_4 = (f_3 + f_4 + f_5)/3$$
$$g_5 = (f_4 + f_5 + f_6)/3$$
$$\vdots$$
$$g_i = (f_{i-1} + f_i + f_{i+1})/3$$

$$\boxed{g_i = \sum_{m'} w_{m'} f_{i+m'} = \sum_m w_m f_{i-m}}$$

# 5.2 Relation between Moving Average and Convolution

Moving Average≡Discrete Convolution Integral

$g(x)$: averaged, $f(x)$: observed

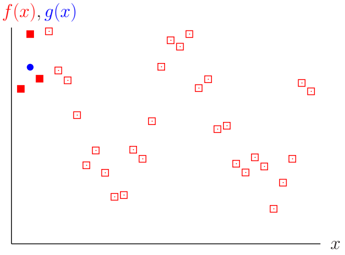
Continuous system: $$g(x) = \int_{-\infty}^{\infty} w(x')f(x-x')\,dx' \quad \text{(Convolution)}$$

Discrete system: $$g_i = \sum_{m=-\infty}^{\infty} w_m f_{i-m} \Delta x \qquad (f_i \equiv f(x_i))$$

If $m$ is finite $(N_m = 2N+1)$,

$$g_i = \sum_{m=-N}^{N} \widehat{w}_m f_{i-m} \quad (\widehat{w}_m = w_m \Delta x) \quad (1)$$

$$\int_{-\infty}^{\infty} w(x')\,dx' = 1 \quad \text{(Normalization)} \quad (2)$$



# 5.3 Simple Moving Average

Weight $w(x')$ inside the window is constant

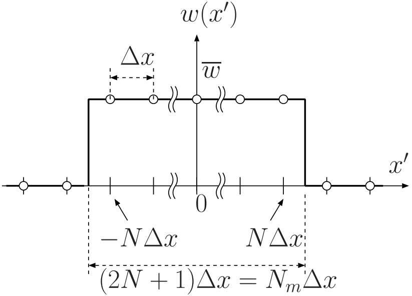
$$w(x') = \begin{cases} \text{const} = \overline{w} & \left(|x'| \leq \dfrac{N_m \Delta x}{2}\right) \\ 0 & \text{(elsewhere)} \end{cases}$$

From Eq.(2)

$$\int_{-\infty}^{\infty} w(x')\,dx' = N_m \Delta x \overline{w}$$

$$= N_m \widehat{w} = 1$$

$$\widehat{w}_m = \begin{cases} \dfrac{1}{N_m} & (|m| \leq N) \\ 0 & \text{(otherwise)} \end{cases}$$



# Comparison of Noise Reduction (original signal)

Before averaging $f_i = \tilde{f}_i + n_i$

($f$: obs., $\tilde{f}$: true (unique), $n$: noise)

(Measurement infinite times: $f_i^j \quad (j \in \{1, \cdots, \infty\})$)

- Average with respect to $j$:

$$\mathrm{E}[f_i] = \mathrm{E}\left[\tilde{f}_i + n_i\right] = \underbrace{\mathrm{E}\left[\tilde{f}_i\right]}_{=\tilde{f}_i} + \underbrace{\mathrm{E}[n_i]}_{=0} = \tilde{f}_i$$

- Variance:

$$\begin{aligned}\sigma_{f_i}^2 &= \mathrm{E}\left[(f_i - \mathrm{E}[f_i])^2\right] \\ &= \mathrm{E}\left[((\tilde{f}_i + n_i) - \tilde{f}_i)^2\right] \\ &= \mathrm{E}\left[n_i^2\right] = \sigma_{n_i}^2 = \sigma_n^2\end{aligned}$$

(Another method to evaluate variance

If two signals $a$ and $b$ have no correlations,

$$\sigma_{a\pm b}^2 = \sigma_a^2 + \sigma_b^2.$$

$\tilde{f}_i$ and $n_i$ have no correlations, and $\sigma_{\tilde{f}_i}^2 = 0$ because $\tilde{f}$ is unique.

$\therefore \sigma_{f_i}^2 = \sigma_{\tilde{f}_i}^2 + \sigma_{n_i}^2 = \sigma_n^2$ )

# Comparison of Noise Reduction (Moving Average)(1)

Moving Average $$g_i = \frac{1}{N_m} \sum_m f_{i-m} \equiv \langle f_i \rangle_m$$

$$f_i = \tilde{f}_i + n_i \rightarrow g_i = \left\langle \tilde{f}_i \right\rangle_m + \langle n_i \rangle_m$$

- Average with respect to $j$:

$$\mathrm{E}\left[g_i\right] = \mathrm{E}\left[\left\langle \tilde{f}_i \right\rangle_m\right] + \mathrm{E}\left[\langle n_i \rangle_m\right] = \left\langle \underbrace{\mathrm{E}\left[\tilde{f}_i\right]}_{=\tilde{f}_i} \right\rangle_m + \left\langle \underbrace{\mathrm{E}\left[n_i\right]}_{=0} \right\rangle_m = \left\langle \tilde{f}_i \right\rangle_m$$

- Variance:

$$\sigma_{g_i}^2 = \mathrm{E}\left[\left(g_i - \mathrm{E}\left[g_i\right]\right)^2\right] = \mathrm{E}\left[\left\{\left(\left\langle \tilde{f}_i \right\rangle_m + \langle n_i \rangle_m\right) - \left\langle \tilde{f}_i \right\rangle_m\right\}^2\right] = \mathrm{E}\left[\langle n_i \rangle_m^2\right]$$

$$= \mathrm{E}\left[\left(\frac{\sum_m n_{i-m}}{N_m}\right) \cdot \left(\frac{\sum_m n_{i-m}}{N_m}\right)\right] = \mathrm{E}\left[\frac{\sum_m \sum_{m'} n_{i-m} n_{i-m'}}{N_m^2}\right]$$

$$= \frac{1}{N_m^2} \sum_m \sum_{m'} \mathrm{E}\left[n_{i-m} n_{i-m'}\right]$$

The variance depends on the auto-correlation of noise.

# Comparison of Noise Reduction (Moving Average)(2)

## Variation depending noise property

$$\sigma_{g_i}^2 = \frac{1}{N_m^2} \sum_m \sum_{m'} \mathrm{E}\left[n_{i-m} n_{i-m'}\right]$$

- Case of White noise:
  ($n_{i-m}$ and $n_{i-m'}$ are independent.)

$$\mathrm{E}\left[n_{i-m} n_{i-m'}\right] = \sigma_n^2 \delta_{m,m'}$$

$$\sigma_{g_i}^2 = \frac{\sigma_n^2}{N_m^2} \underbrace{\sum_m \sum_{m'} \delta_{m,m'}}_{N_m} = \frac{\sigma_n^2}{N_m}$$

- Case of Low frequency noise
  ($n_{i-m} \simeq n_{i-m'}$)

$$\mathrm{E}\left[n_{i-m} n_{i-m'}\right] \simeq \sigma_n^2$$

$$\sigma_{g_i}^2 \simeq \frac{\sigma_n^2}{N_m^2} \underbrace{\sum_m \sum_{m'} 1}_{N_m^2} = \sigma_n^2$$

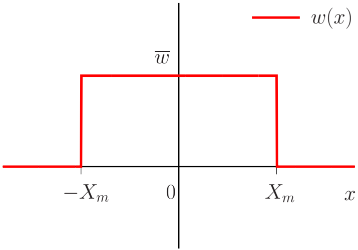
# Comparison of Noise Reduction (Summary)

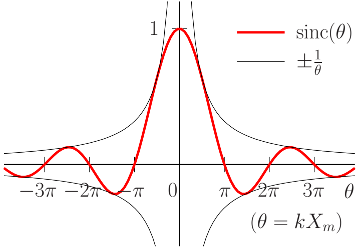
| | Original | Moving Average | |
|---|---|---|---|
| Ave. | $\tilde{f}_i$ | $\left\langle \tilde{f}_i \right\rangle_m$ (Smoothed True signal (drawback)) | |
| Var. | $\sigma_n^2$ | White Noise (Best)<br>$\dfrac{\sigma_n^2}{N_m}$<br>reduced to $1/N_m$times (advantage) | Low frequency noise (Worst)<br>$\sigma_n^2$<br>Not reduced |

# Spectral Gain of simple moving average

Fourier Transform of $w(x')$

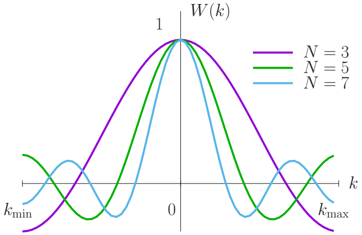
$$w(x') = \overline{w} \equiv \frac{1}{2X_m}$$
$$\left(|x'| \leq X_m = \frac{N_m \Delta x}{2}\right)$$
$$W(k) = \int_{-\infty}^{\infty} w(x') e^{-ikx'}\, dx'$$
$$= \frac{1}{2X_m} \int_{-X_m}^{X_m} e^{-ikx'}\, dx'$$
$$= \frac{1}{-2ikX_m}\left(e^{-ikX_m} - e^{+ikX_m}\right)$$
$$= \frac{\sin(kX_m)}{kX_m} = \mathrm{sinc}(kX_m)$$



# Spectral Gain of simple moving average (cont.)

$$g(x) = w(x) * f(x)$$
$$G(k) = W(k) \cdot F(k)$$
$$W(k) = \mathrm{sinc}(kX_m)$$
$$(X_m = \frac{N_m \Delta x}{2})$$




Moving average can filtrate higher frequency components.

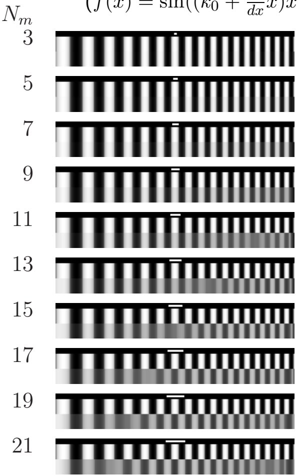
However,
$$G(k) = W(k)F(k) = W(k)\left[\widetilde{F}(k) + N(k)\right]$$
$$= W(k)\widetilde{F}(k) + W(k)N(k)$$

Since the higher frequency components of the true signal, $\tilde{f}(x)$, are reduced as well as those of the noise ($n(x)$), the true signal is distorted.

# 5.4 Distortion and Spurious Resolution

Distortion of true signal by moving average

$(f(x) = \sin((k_0 + \frac{dk}{dx}x)x),\ \sigma_n^2 = 0)$




(Contrast is enhanced)

- With increasing number of sampling $N_m$, intensity of higher frequency components become smaller.

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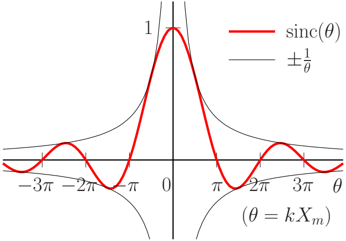
  The choice of appropriate $N_m$ is important.
- When $N_m$ is large, intensity becomes larger at higher frequency but those patterns are inverted. (Spurious resolution)

# Cause of Spurious Resolution




With increasing $k$, $W(k)$ is reduced.
$W(k)$ becomes 0 at $kX_m = \pi$.
After that it becomes negative.
$W(k) < 0 \rightarrow$ Inversion of intensity.

The spurious resolution is found in other filtering.