3.6 Generation of Random Noise

Ideal Random Number: No periodicity $\rightarrow$ White Noise

- Computer simulation: Random numbers are required for generation of simulated noise. Normal rand. num. are required in many cases.
- A function in most computers is only a series of Uniform Distrib.
  Uniform Random number: $X \sim U[\min, \max]$
  Implemented one in computer system is $X \sim U[0, 1]$.
  ($X \in [0, 1]$ is generated and the probability to generating them is same.)
- Methods to generate Normal Rand. from Uniform Rand.
  - Sum of plural Uniform Random numbers.
  - Coordinate transform of Uniform Random numbers
  - Use of Multi-dimensional probability distribution function

Probability of independent events

- Random Variable: X
  $X \in \{x_1, x_2, \cdots, x_n\}$ (Discrete)
  $X \in [x_{\min}, x_{\max}]$ (Continuous)
- Probability Density Function: $p(x)$
  Probability of $X = x$:
  (In cont. system, $p(x)\,dx$.)
- Cumulative Distribution Function:
  $$P(X < x) = \int_{-\infty}^{x} p(x)\,dx$$
  $$(P(X < \infty) = 1)$$
- Average (expected value) of X:
  $$\overline{x} = \int_{-\infty}^{\infty} x p(x)\,dx$$
- Average of function with X:
  $$\overline{f(X)} = \mathrm{E}[f(X)] = \int_{-\infty}^{\infty} f(x)p(x)\,dx$$

- Variance of X:
  $$\sigma_x^2 = \overline{(x-\overline{x})^2} = \overline{x^2} - \overline{x}^2$$
  $$\left(\begin{aligned} \overline{(x-\overline{x})^2} &= \int (x-\overline{x})^2\,p(x)\,dx \\ &= \int (x^2 - 2x\overline{x} + \overline{x}^2)\,p(x)\,dx \\ &= \underbrace{\int x^2\,p(x)\,dx}_{=\overline{x^2}} - 2\overline{x}\underbrace{\int x\,p(x)\,dx}_{=\overline{x}} + \overline{x}^2\underbrace{\int p(x)\,dx}_{=1} \\ &= \overline{x^2} - \overline{x}^2 \end{aligned}\right)$$
- Incident Prob. Dens. Func. for two independent events:
  $$p(x_1, x_2) = p_1(x_1)\,p_2(x_2)$$
  - $\overline{x_1 + x_2} = \overline{x_1} + \overline{x_2}$
  - $\sigma^2_{x_1+x_2} = \sigma^2_{x_1} + \sigma^2_{x_2}$
  - $\overline{x_1 x_2} = \overline{x_1} \cdot \overline{x_2}$

Generation of Normal random distribution using sum of uniform random numbers

Central Limit Theorem

Sum of Random Variables which obey an independent distribution converges to a Normal distribution. (There are some exceptions.)

- $X_u \sim U[0,1] \quad \rightarrow \quad p(x) = 1$
- Sum of two random number:

$$F(x) = P\left(X_1 + X_2 < x\right) = \iint_{x_1+x_2<x} dx_1\, dx_2$$

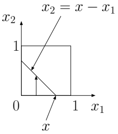
Case of $0 < x \leq 1$:

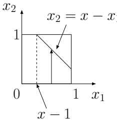
$$F(x) = \int_{x_1=0}^{x} \int_{x_2=0}^{x-x_1} dx_2\, dx_1 = \frac{x^2}{2}$$

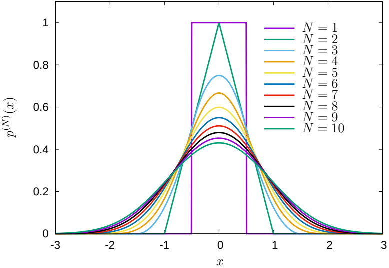
Case of $1 < x \leq 2$:

$$F(x) = \left(\int_{x_1=0}^{x-1} \int_{x_2=0}^{1} + \int_{x_1=x-1}^{1} \int_{x_2=0}^{x-x_1} \right) dx_2\, dx_1$$
$$= -\frac{x^2}{2} + 2x - 1$$

$$p(x) = \begin{cases} x & (0 < x \leq 1) \\ 2 - x & (1 < x \leq 2) \end{cases}$$







- $X_u \sim U[0,1] \quad \rightarrow \quad p(x) = 1$
- Average and Variance

$$\begin{cases} \overline{x_u} = \int_0^1 x_u \, dx_u = \dfrac{1}{2} \\ \sigma_{x_u}^2 = \int_0^1 (x_u - \overline{x_u})^2 \, dx_u = \overline{x_u^2} - \overline{x_u}^2 \\ \qquad = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \end{cases}$$

- $X = \sum_{n=1}^{N} X_u$

$$\begin{cases} \overline{x} = \sum_{n=1}^{N} \overline{x_u} = \dfrac{N}{2} \\ \sigma_x^2 = \sum_{n=1}^{N} \sigma_{x_u}^2 = \dfrac{N}{12} \end{cases}$$

When $N = 12$,

$$X = \sum_{n=1}^{12} X_u - 6 \sim N[0,1]$$

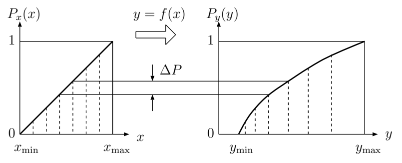
is obtained as a Normal distribution.

($N[\bar{x}, \sigma_x^2]$:
Normal distribution
with average, $\bar{x}$
with variance, σ_x^2)

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{(x-\bar{x})^2}{2\sigma_x^2}}$$

Arbitrary random distribution using coordinate transform

Coordinate change using $y = f(x)$ from $P_x(X < x)$ to $P_y(Y < y)$.



$\Delta P = p_x(x)\,dx$ $\Delta P = p_y(y)\,dy$

$$p_x(x)\,dx = p_y(y)\,dy$$

We can obtain the transformation $f(x)$ from $X \sim U[0,1]$ $(p_x(x) = 1)$ so that $p_y(y)$ satisfies above the Eq.

- Exponential distribution

$$p_y(y) = e^{-y}$$
$$dx = e^{-y}\,dy$$
$$\rightarrow x = -e^{-y} \rightarrow y = -\log x$$

> The exponential random distribution can obtained by transform with $y = -\log x$ where x is uniform random number.

- Randoms obeying Normal dist.

$$p_y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$$
$$x = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{y^2}{2}}\,dy = \frac{1 + \mathrm{erf}(\frac{y}{2})}{2}$$
$$\rightarrow y = 2\,\mathrm{erf}^{-1}(2x - 1)$$

erf^{-1} is not implemented in computers.

Normal random number using Box-Muller method

Multi-dimensional probability distribution:

$$
\begin{aligned}
p_y(y_1, y_2, \cdots)\, dy_1\, dy_2 \cdots &= p_x(x_1, x_2, \cdots)\, dx_1\, dx_2 \cdots \\
&= p_x(x_1, x_2, \cdots)|J|\, dy_1\, dy_2 \cdots
\end{aligned}
$$

Assume x_1, x_2 are independent.

$X_1, X_2 \sim U[0,1]$

$(p_x(x_1, x_2) = p_x(x_1)p_x(x_2) = 1)$

$$
\begin{cases} y_1 = \sqrt{-2\log x_1}\cos(2\pi x_2) \\ y_2 = \sqrt{-2\log x_1}\sin(2\pi x_2) \end{cases} \qquad (1)
$$

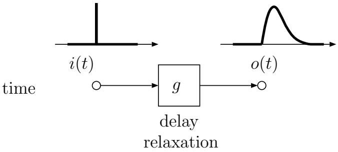
x_1 and x_2 are

$$
\begin{cases} x_1 = e^{-\frac{y_1^2 + y_2^2}{2}} \\ x_2 = \frac{1}{2\pi}\tan^{-1}\left(\frac{y_2}{y_1}\right) \end{cases}
$$

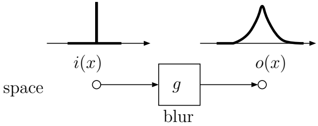
$$
\begin{aligned}
|J| &= \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \frac{1}{2\pi} e^{-\frac{y_1^2 + y_2^2}{2}} \\
&= \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{y_1^2}{2}}\right)\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{y_2^2}{2}}\right) \\
&= p_y(y_1)p_y(y_2)
\end{aligned}
$$

The pair (y_1, y_2) has Normal distribution with $N[0,1]$, which is obtained by Eq. (1) using a pair of (x_1, x_2) with $U[0,1]$.

4. Convolution and Response Function



$$o(t) = \int_0^{\infty} g(\tau) i(t-\tau)\, d\tau$$
$$= \int_{-\infty}^{t} i(\tau) g(t-\tau)\, d\tau \equiv g(t) * i(t) \equiv (g * i)(t)$$



$$o(x) = \int_{-\infty}^{\infty} g(\xi) i(x-\xi)\, d\xi$$
$$= \int_{-\infty}^{\infty} i(\xi) g(x-\xi)\, d\xi \equiv g(x) * i(x) \equiv (g * i)(x)$$

- Since input is considered as reason, and output as result, the output is considered as integral of the input.
- The response does not depend on absolute time. It only depend on the time difference between input and output.
- The output is an integral of the product between input and weight depending time difference.
- The difference of time and spatial domain is only the region of integral. (Causality)

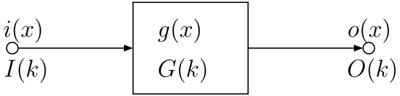
  ※ The convolution is similar to the cross-correlation but sign of the argument is inverted.

Convolution theorem

4.1 Convolution theorem

Convolution theorem

If $o(x)$ is represented as a convolution of $g(x)$ and $i(x)$,



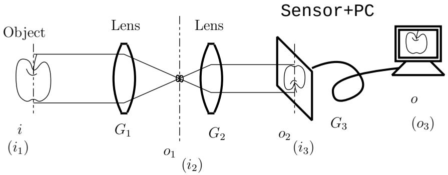
$$o(x) = g(x) * i(x)$$
$$O(k) = G(k) \cdot I(k)$$

$$\begin{cases} F(k) = \displaystyle\int f(x)\, e^{-ikx}\, dx \\ f(x) = \dfrac{1}{2\pi} \displaystyle\int F(k)\, e^{+ikx}\, dk \end{cases}$$

$$o(x) = \int_\xi g(\xi)\, i(x-\xi)\, d\xi$$

$$\begin{aligned} O(k) &= \int_x o(x)\, e^{-ikx}\, dx \\ &= \int_x \left(\int_\xi g(\xi)\, i(x-\xi)\, d\xi \right) e^{-ikx}\, dx \\ &= \int_\xi g(\xi) \left(\int_x i(x-\xi)\, e^{-ikx}\, dx \right) d\xi \\ &= \int_\xi g(\xi) \left(\int_x i(x-\xi)\, e^{-ik(x-\xi)}\, dx\ e^{-ik\xi} \right) d\xi \\ &\qquad\qquad (x' = x - \xi) \\ &= \left(\int_\xi g(\xi)\, e^{-ik\xi}\, d\xi \right) \cdot \left(\int_{x'} i(x')\, e^{-ikx'}\, dx' \right) \\ &= G(k) \cdot I(k) \end{aligned}$$

4.2 Response Function



$$o_n(\boldsymbol{r}) = \int g_n(\boldsymbol{\xi})\, i_n(\boldsymbol{r}-\boldsymbol{\xi})\, d\boldsymbol{\xi}$$
$$= (g_n * i_n)(\boldsymbol{r}) = i_{n+1}(\boldsymbol{r})$$
$$O_n(\boldsymbol{k}) = G_n(\boldsymbol{k})\, I_n(\boldsymbol{k}) = I_{n+1}(\boldsymbol{k})$$

$$o(\boldsymbol{r}) = (g_N * (\cdots * (g_1 * i)))(\boldsymbol{r}), \qquad O(\boldsymbol{k}) = \left(\prod_{n=1}^{N} G_N(\boldsymbol{k})\right) I(\boldsymbol{k})$$

- The response function of a whole system equals to the products of response functions of each components.
- When the response function in each system is known, $i(\boldsymbol{r})$ can be obtained from $o(\boldsymbol{r})$.

Measurement of Response function

- Response func. of pin-hole $(i(\boldsymbol{r}) = \delta(\boldsymbol{r}))$

$$o(\boldsymbol{r}) = \int g(\boldsymbol{r} - \boldsymbol{\xi})\, \delta(\boldsymbol{\xi})\, d\boldsymbol{\xi} = g(\boldsymbol{r})$$
$$O(\boldsymbol{k}) = G(\boldsymbol{k})$$

  $o(\boldsymbol{r})$: Point Spread Function
  $O(\boldsymbol{k})$: Point Response Function

- Response func. of slit $(i(\boldsymbol{r}) = \delta(x))$
  $o_x(\boldsymbol{r})$: Line Spread Function
  $O_x(\boldsymbol{k})$: Line Response Function

Is there an ideal pin hole or slit?

- Response of edge $(i_e(\boldsymbol{r}) = \theta(x))$

$(\theta$: step func., $\frac{d\theta}{dx} = \delta(x))$

$$o_e(x) = \int g_x(x - \xi)\, \theta(\xi)\, d\xi = \int g_x(\xi')\, \theta(x - \xi')\, d\xi'$$
$$\frac{do_e(x)}{dx} = \frac{d}{dx} \int g_x(\xi')\, \theta(x - \xi')\, d\xi'$$
$$= \int g_x(\xi')\, \underbrace{\frac{d\theta(x - \xi')}{d(x - \xi')}}_{\delta(x-\xi')}\, \underbrace{\frac{d(x - \xi')}{dx}}_{=1}\, d\xi'$$
$$= \int g_x(\xi')\, \delta(x - \xi')\, d\xi' = g_x(x)$$
$$G_x(k_x) = \int g_x(x)\, e^{-ik_x x}\, dx = \int \frac{do_e}{dx}\, e^{-ik_x x}\, dx$$
$$= \underbrace{\left[o_e(x)\, e^{-ik_x x}\right]_{-\infty}^{\infty}}_{=0 \quad (\because o_e(\pm\infty)=0)} + ik_x \int o_e(x)\, e^{-ik_x x}\, dx$$
$$= ik_x O_e(k_x)$$

$O_e(\boldsymbol{k})$: Edge Response Function

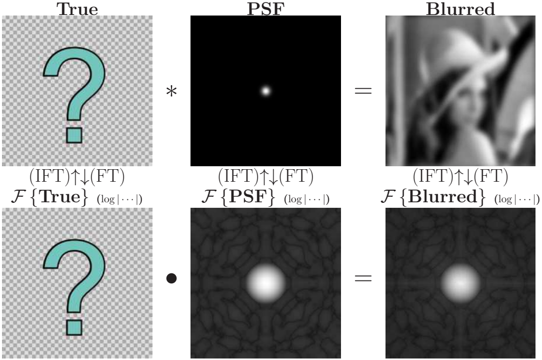
 Deconvolution

4.3 Deconvolution

A blurred image and the PSF of whole system is known.

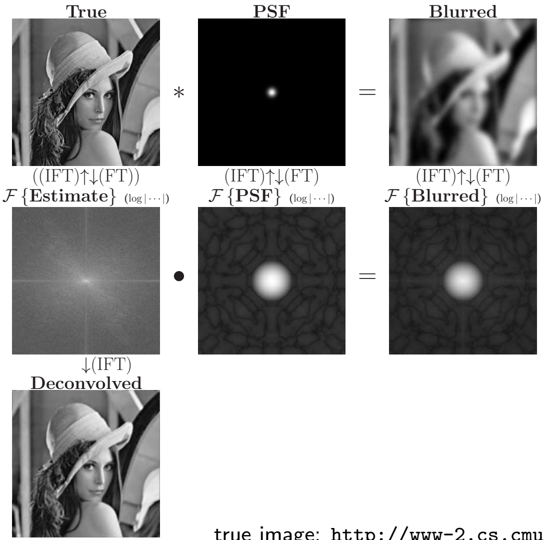
$\rightarrow \mathcal{F}\{\text{Blurred}\}$ and $\mathcal{F}\{\text{PSF}\}$ are also known.

We wish to reconstruct the true image.



$$\text{True'} = \mathcal{F}^{-1}\left\{\frac{\mathcal{F}\{\text{Blurred}\}}{\mathcal{F}\{\text{PSF}\}}\right\} \quad \text{(Deconvolution)}$$

Example of reconstruction from blurred image



In $\mathcal{F}\{\text{Blurred}\}/\mathcal{F}\{\text{PSF}\}$, it has a divergence if the denominator is small.

↓

Since special treatments are applied (discuss in later days) to avoid divergence, True $\neq$ Deconvoluted

※ In this example, it reconstructed only from the Blurred image, i.e., PSF is also unknown.

true image: `http://www-2.cs.cmu.edu/~chuck/lennapg/lena_std.tif`