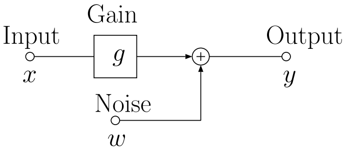
Observation Model of Additive Noise

3. Noise
3.1 Observation Model of Additive Noise

Model of Additive Noise



$$y = gx + w$$

N times measurements

$$(y^{(n)}, n \in \{1, \cdots, N\})$$

$$y^{(n)} = gx^{(n)} + w^{(n)}$$

$$\left(\begin{array}{cl} y^{(n)} & : \text{ known} \\ x^{(n)}, w^{(n)} & : \text{ unknown} \end{array} \right)$$

- x and w are independent
- Definition of Expected value and Variance

  Expected Value :

  $$\mathrm{E}[f] \equiv \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f^{(n)} \equiv \overline{f}$$

  Variance :

  $$\sigma_f^2 \equiv \mathrm{E}\left[\left(f - \overline{f}\right)^2\right]$$

- What is the relation between $\overline{x}$, $\overline{w}$, and $\overline{y}$, or that between σ_x^2, σ_w^2, and σ_y^2?

Statistics of Observed value in the additive noise model

$y = gx + w$

- Ave. (Exp. Val.)

$$\overline{y} = \mathrm{E}\left[y\right] = \mathrm{E}\left[gx + w\right] = \lim_{N \to \infty} \frac{1}{N} \sum \left(gx^{(n)} + w^{(n)} \right) = g\mathrm{E}\left[x\right] + \mathrm{E}\left[w\right] = g\overline{x} + \overline{w}$$

- Var.

$$\sigma_y^2 = \mathrm{E}\left[\left((gx+w) - \overline{y}\right)^2\right] = \mathrm{E}\left[\left((gx+w) - (g\overline{x} + \overline{w})\right)^2\right] = \mathrm{E}\left[\left(g(x-\overline{x}) + (w - \overline{w})\right)^2\right]$$

$$= g^2 \underbrace{\mathrm{E}\left[(x-\overline{x})^2\right]}_{=\sigma_x^2} + 2g \underbrace{\mathrm{E}\left[(x-\overline{x})(w-\overline{w})\right]}_{=0\ (\because \mathrm{E}[xw] = \overline{x} \cdot \overline{w})} + \underbrace{\mathrm{E}\left[(w-\overline{w})^2\right]}_{=\sigma_w^2} = g^2 \sigma_x^2 + \sigma_w^2$$

> In general, $\overline{w} = 0$, $\sigma_w^2 > 0$. When $N \to \infty$,
>
> - $\overline{y} = g\overline{x}$
> $\rightarrow$ Effect of noise can be removed.
> - $\sigma_y^2 = g^2\sigma_x^2 + \sigma_w^2 > g^2\sigma_x^2$
> $\rightarrow$ The variance caused by noise cannot be removed.

3.2 Classification of random signal

- Stationary
  - Ergodic
  - Non-Ergodic
- Not stationary
  - Non-ergodic

- Multiple measurements of time varying signal $(f^{(n)}(t),\ n \in \{1, \cdots, N\})$

$$\overline{f}(t) = E[f(t)] = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f^{(n)}(t)$$

$$C(t,\tau) = E[f(t)f(t+\tau)]$$

$$= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f^{(n)}(t) f^{(n)}(t+\tau)$$

  - Stationary :
    $\overline{f}(t) = \overline{f'}$, $C(t,\tau) = C'(\tau)$
    (Independent of t)

- One of time varying signal

$$\langle f \rangle^{(n)} = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f^{(n)}(t)\, dt$$

$$C^{(n)}(\tau)$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f^{(n)}(t) f^{(n)}(t+\tau)\, dt$$

  - Ergodic :
    $\langle f \rangle^{(n)} = \overline{f'}$, $C^{(n)}(\tau) = C'(\tau)$

3.3 Spectrum of Noise White Noise

Stationary Ergodic random signal

$$(\langle n(t)\rangle = 0, \sigma_n^2 = \left\langle n^2(t)\right\rangle = \overline{n^2} > 0)$$

White Noise : Independent at $\tau \neq 0$.

- Auto correlation : $$C(\tau) = \langle n(t)n(t+\tau)\rangle = \lim_{\Delta t \to 0} \overline{n^2}\delta(\tau)\Delta t \quad (1)$$

  (Δt is added to match the dimension.)

- Power Spectrum : $$S(\omega) = \int \overline{n^2}\delta(\tau)\Delta t e^{-j\omega\tau}\, d\tau = \overline{n^2}\Delta t \quad (2)$$

  $\rightarrow$ Spectrum is constant.$\Leftrightarrow$White

- Cross correlations between other signals are 0.

$$C_{nf}(\tau) = \langle n(t)f(t+\tau)\rangle = \underbrace{\langle n(t)\rangle}_{=0}\langle f(t+\tau)\rangle = 0 \qquad (3)$$

Brownian Noise($1/f^2$ Noise)

- Markov Process(Affected by the previous point(neighbors))

$$r(t+\Delta t) = \rho r(t) + n(t) \quad (4)$$

$\left(0 < \rho < 1,\ \Delta t > 0,\ n(t) :\right.$ white noise $\left.\right)$

- Auto correlation

$$C(\tau) = \langle r(t) r(t+\tau) \rangle$$

$$\begin{aligned} C(\tau + \Delta t) &= \langle r(t) r(t+\tau+\Delta t) \rangle \\ &= \langle r(t) \{\rho r(t+\tau) + n(t+\tau)\} \rangle \\ &= \rho \langle r(t) r(t+\tau) \rangle + \langle r(t) n(t+\tau) \rangle \\ &= \rho C(\tau) \end{aligned}$$

Compare Taylor series expansion.$(\tau > 0)$

$$C(\tau + \Delta t) = C(\tau) + \frac{dC}{d\tau}\Delta t + O(\Delta t^2)$$

$$\rightarrow \quad \rho C = C + \frac{dC}{d\tau}\Delta t$$

$$\rightarrow \quad \frac{1}{C} dC = -\underbrace{\frac{1-\rho}{\Delta t}}_{=\alpha} d\tau = -\alpha\, d\tau$$

$$C(\tau) = C_0 e^{-\alpha\tau}$$

Since $C(\tau)$ has an even property,

$$\boxed{C(\tau) = C_0 e^{-\alpha|\tau|} \quad (5)}$$

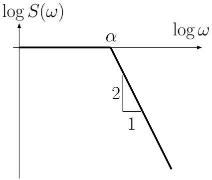
C_0 is obtained from Eq.(4).

$$\langle r^2(t+\Delta t) \rangle = \langle (\rho r(t) + n(t))^2 \rangle$$

$$C_0 = \rho^2 C_0 + \sigma_n^2$$

$$\therefore \quad C_0 = \frac{\sigma_n^2}{1-\rho^2} \quad (6)$$

- Power spectrum



$$\begin{aligned} S(\omega) &= \int_{-\infty}^{0} C_0 e^{(\alpha - j\omega)t}\, dt + \int_{0}^{\infty} C_0 e^{(-\alpha - j\omega)t}\, dt \\ &= C_0 \left(\frac{1}{\alpha - j\omega} + \frac{-1}{-\alpha - j\omega} \right) = \frac{2C_0\alpha}{\omega^2 + \alpha^2} \end{aligned} \quad (7)$$

$$\text{Lorentz distribution} \propto \frac{1}{\omega^2 + \alpha^2}$$

$$S(\omega) \propto \begin{cases} \text{const} & (\omega \ll \alpha) \\ \frac{1}{\omega^2} \propto \frac{1}{f^2} & (\omega \gg \alpha) \end{cases} \quad (8)$$

- Naming
  - Red Noise ←Higher freq. (short wavelength) comp. is small.
  - Brownian noise (Not color 'brown')
    ←Spectrum of particle position with Brownian motion (Random walk)
  - Lorentzian Noise
- e.g. Thermal noise

$1/f$ Noise

Noise with $S(\omega) \propto \frac{1}{\omega}$ $\quad$ $(※\ |X(\omega)| \propto \frac{1}{\sqrt{\omega}} \neq \frac{1}{\omega})$

Signals with $1/f$ noise

- Electric resistance of metal (fluctuation of num. of carriers)
- Sound from small stream of water
- pitch of grain of wood

The mechanism is not known clearly.

(Another name) Pink noise (Intermediate White and Red.)

3.4 Auto correlation of observed value.

$s'(t) = s(t) + n(t)$

$s(t)$	: True(Unknown.)	$C_s(\tau)$ is also unknown.
$n(t)$	: White Noise(Unknown.)	$C_n(\tau) = \overline{n^2}\delta(\tau)\Delta t$ is known.
$s'(t)$	: Observed(Known.)	$C_{s'}(\tau)$ is also known.

$$\begin{aligned} C_{s'}(\tau) &= \left\langle s'(t)s'(t+\tau)\right\rangle = \langle\{s(t)+n(t)\}\{s(t+\tau)+n(t+\tau)\}\rangle \\ &= \underbrace{\langle s(t)s(t+\tau)\rangle}_{=C_s(\tau)} + \underbrace{\langle n(t)s(t+\tau)\rangle}_{=0} + \underbrace{\langle s(t)n(t+\tau)\rangle}_{=0} + \underbrace{\langle n(t)n(t+\tau)\rangle}_{=C_n(\tau)} \\ &= C_s(\tau) + \overline{n^2}\delta(\tau)\Delta t \end{aligned}$$

$$\therefore C_s(\tau) = C_s'(\tau) - \overline{n^2}\delta(\tau)\Delta t$$

If we know property of noise, we can obtain auto correlation of true.

3.5 Intensity distribution of Noise

- The terminology White or Red expresses spectrum in freq. domain. This represents periodicity of signal, it does not represent the intensity distribution of noise.
- The quantity to represent the intensity is probability distribution p. (Form of the function and Parameters (eg. standard deviation).
- Well-used probability distribution :
  - Normal distribution (Gaussian distribution)
  - Uniform distribution
- To express the property of noise, both the spectrum and distribution function are required.

  (eg. White Normal distributed noise (with standard deviation))