2. Power Spectrum and Correlation function

- What is the correlation?
  Relation between two discrete data set $(x_i,\, y_i)$
- What is the correlation function?
  Relation between two variates which are represented as continuous function $(x(t),\, y(t))$
- What is the power spectrum?
  Measure of Fourier transformed function $X(\omega)$ of a variate $x(t)$.
- Wiener-Khintchine's theorem
  Relation between power spectrum and auto-correlation function

2.1 Definition of Power Spectrum

Fourier Transform :

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega \qquad (1)$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt \qquad (2)$$

Power spectral density

$$S(\omega) = \lim_{T\to\infty}\frac{|X(\omega)|^2}{T} \qquad (3)$$

- Time average of the square of sinusoidal component with frequency ω in the signal.
- No information about phase.
- $S(\omega)\,d\omega$ expresses the power spectrum.
- $|X(\omega)|^2$ is called energy spectrum.
- Dimension analysis:

FT of x	:	$[X] = [x \cdot t]$
Power spectral density	:	$[S] = \left[\frac{\|X\|^2}{T}\right] = [x^2 \cdot t]$
Power spectrum	:	$[S\,d\omega] = \left[\frac{X^2}{t}\frac{1}{t}\right] = [x^2]$
Energy spectral density	:	$[\|X\|^2] = [x^2 t^2]$

2.2 Correlation

Population: Sample f_i $(i \in (1, \cdots, N))$

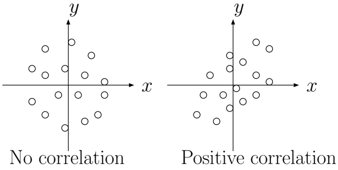
Average $$\mathrm{E}[f] \equiv \frac{1}{N}\sum_{i=1}^{N} f_i \qquad (4)$$

Variance $$\sigma_f^2 \equiv \mathrm{E}\left[(f - \mathrm{E}[f])^2\right] \qquad (5)$$

Correlation: Index to represent similarity of two variates $(x_i,\ y_i)$.

$$C = \mathrm{E}\left[x'y'\right], \quad \text{or} \quad r = \frac{\mathrm{E}[x'y']}{\sqrt{\mathrm{E}\left[x'^2\right]\mathrm{E}\left[y'^2\right]}} \qquad (6)$$

$$(x_i' = x_i - \mathrm{E}[x], \quad y_i' = y_i - \mathrm{E}[y])$$

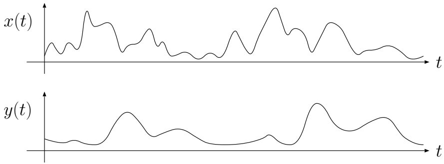


2.3 Correlation Function

In case where x and y are variates with respect to time:

$$\mathrm{E}\left[x\right] \rightarrow \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\, dt = \langle x(t) \rangle_t \quad \text{(Time Average)} \tag{7}$$

e.g. $x(t)$: Amount of rainfall,
$y(t)$: Amount of water in a river



- Time delay
- Smoothing of time fluctuation

Cross-correlation function and Auto-correlation function

Cross-correlation function

$$C_{xy}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)y(t+\tau)\,dt = \langle x(t)y(t+\tau)\rangle_t \qquad (8)$$

Even when $y(t) = x(t)$, we can understand the periodicity of $x(t)$.

Auto-correlation function

$$C(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)x(t+\tau)\,dt = \langle x(t)x(t+\tau)\rangle_t \qquad (9)$$

(Normalization)

$$R(\tau) = \frac{C(\tau)}{C(0)} = \frac{\langle x(t)x(t+\tau)\rangle_t}{\langle x(t)^2\rangle_t}, \qquad R(0) = 1$$

$\rightarrow$ auto-correlation coefficient

Periodicity of auto-correlation function

(e.g. 1) $x(t) = A\cos\omega_1 t$

$$C(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}(A\cos\omega_1 t)(A\cos\omega_1(t+\tau))\,dt$$

$$= A^2\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\left(\cos^2\omega_1 t\cos\omega_1\tau - \cos\omega_1 t\sin\omega_1 t\sin\omega_1\tau\right)\,dt$$

$$= A^2\lim_{T\to\infty}\Bigg[\underbrace{\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\frac{1+\cos 2\omega_1 t}{2}\,dt}_{=\frac{1}{2}}\cos\omega_1\tau - \underbrace{\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\frac{\sin 2\omega_1 t}{2}\,dt}_{=0}\sin\omega_1\tau\Bigg] = \frac{A^2}{2}\cos\omega_1\tau$$

$$\underline{R(\tau) = \cos\omega_1\tau}$$

(e.g. 2) $x(t) = A\sin\omega_1 t = A\cos\left(\omega_1 t - \frac{\pi}{2}\right)$

$$C(\tau) = A^2\lim_{T\to\infty}\left[\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\frac{1+\cos\left(2\left(\omega_1 t - \frac{\pi}{2}\right)\right)}{2}\,dt\cos\omega_1\tau - \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\frac{\sin\left(2\left(\omega_1 t-\frac{\pi}{2}\right)\right)}{2}\,dt\sin\omega_1\tau\right] = \frac{A^2}{2}\cos\omega_1\tau$$

$$\underline{R(\tau) = \cos\omega_1\tau}$$

- Periodicity is found.
- Independent of phase $\rightarrow$ Independent of the origin of t.

Characteristics of Auto-correlation function

- Independent of the position of origin.
- Even function. $(C(-\tau) = C(\tau))$

$$\left(\begin{aligned} C(-\tau) &= \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)x(t-\tau)\,dt \qquad (t' = t-\tau) \\ &= \lim_{T\to\infty} \int_{-\frac{T}{2}-\tau}^{\frac{T}{2}-\tau} x(t'+\tau)x(t')\,dt' \qquad \left(\frac{T}{2} \gg |\tau|\right) \\ &= \lim_{T\to\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t'+\tau)x(t')\,dt' = C(\tau) \end{aligned} \right)$$

- Maximum at $\tau = 0$.

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} (x(t) \pm x(t+\tau))^2 \; dt \geq 0$$

$$\left(\begin{aligned} \text{LHS} &= \underbrace{\int x^2(t)\,dt}_{C(0)\geq 0} + \underbrace{\int x^2(t+\tau)\,dt}_{C(0)\geq 0} \pm 2 \underbrace{\int x(t)x(t+\tau)\,dt}_{C(\tau)} \\ &= 2(C(0) \pm C(\tau)) \geq 0 = \text{RHS} \\ &\therefore C(0) \geq |C(\tau)| \end{aligned} \right)$$

- The 1st differential of $C(\tau)$

$$\left(\begin{aligned} C'(\tau) &= \frac{dC}{d\tau} = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\underbrace{\frac{\partial x(t+\tau)}{\partial \tau}}_{=x'(t+\tau)}\,dt = \big\langle x(t)x'(t+\tau)\big\rangle_t \\ &\text{(Replacing } t+\tau=\xi \to dt = d\xi, \text{because } \tfrac{\partial(t+\tau)}{\partial\tau}=1) \\ &= \lim_{T\to\infty}\int_{-\frac{T}{2}+\tau}^{\frac{T}{2}+\tau} x(\xi-\tau)x'(\xi)\,d\xi \\ &= \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(\xi-\tau)x'(\xi)\,d\xi = \big\langle x(t-\tau)x'(t)\big\rangle_t \end{aligned}\right)$$

Cross-correlation between $x(t)$ and $x'(t)$.

- The 2nd differential of $C(\tau)$

$$\left(\begin{aligned} C''(\tau) &= \frac{d^2C}{d\tau^2} = \big\langle x(t)x''(t+\tau)\big\rangle_t \\ &\text{(or replacing } t-\tau=\eta\ (\tfrac{\partial(t-\tau)}{\partial\tau}=-1)) \\ &= \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}}\big(-x'(\eta)x'(\eta+\tau)\big)\,d\eta \\ &= -\big\langle x'(t)x'(t+\tau)\big\rangle_t \end{aligned}\right)$$

Cross-correlation between $x(t)$ and $x''(t)$.

and

Negative of auto-correlation of $x'(t)$.

2.4 Relation between Power spectrum and Auto-correlation

Domain of signal: $(x(t) \in \mathbb{R})$

$$x(t) : \begin{cases} \neq 0 & -\frac{T}{2} \leq t \leq \frac{T}{2} \\ = 0 & \text{elsewhere} \end{cases}$$

Fourier transform (integral):

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, d\omega \left(= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) e^{-j\omega t}\, d\omega \right)$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$$

Auto-correlation function:

$$\begin{aligned} C(\tau) &= \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) x(t+\tau)\, dt \\ &= \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega') e^{-j\omega' t}\, d\omega' \right] \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega(t+\tau)}\, d\omega \right] dt \\ &= \lim_{T\to\infty} \frac{1}{2\pi T} \int_{-\infty}^{\infty} X(\omega) e^{j\omega\tau} \int_{-\infty}^{\infty} X^*(\omega') \underbrace{\left(\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j(\omega-\omega')t}\, dt \right)}_{\delta(\omega-\omega')} d\omega'\, d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{\lim_{T\to\infty} \frac{X^*(\omega) X(\omega)}{T}}_{=S(\omega)} e^{j\omega\tau}\, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) e^{j\omega\tau}\, d\omega \end{aligned}$$

$C(\tau)$ is identical to the inverse Fourier transform of $S(\omega)$.

Wiener-Khintchine's theorem

$$C(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)e^{j\omega\tau}\,d\omega \quad (10)$$

$$S(\omega) = \int_{-\infty}^{\infty} C(\tau)e^{-j\omega\tau}\,d\tau \quad (11)$$

$$\begin{array}{ccc} x(t) & \xrightarrow{\langle x(t)x(t+\tau)\rangle_t} & C(\tau) \\ \downarrow\uparrow \text{FT} & & \downarrow\uparrow \text{FT} \\ X(\omega) & \xrightarrow{\langle X(\omega)X^*(\omega)\rangle_t} & S(\omega) \end{array}$$

$\langle \cdots \rangle_t$: Time average when $T \to \infty$.

2.5 Cross-spectrum and Cross-correlation

$$C_{xy}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xy}(\omega)e^{j\omega\tau}\,d\omega \quad (12)$$

$$S_{xy}(\omega) = \int_{-\infty}^{\infty} C_{xy}(\tau)e^{-j\omega\tau}\,d\tau \quad (13)$$

Cross-spectral density

$$S_{xy}(\omega) = \lim_{T\to\infty}\frac{X^*(\omega)Y(\omega)}{T} = \langle X^*(\omega)Y(\omega)\rangle_t \quad (14)$$

$$\begin{aligned}
C_{xy}(\tau) &= \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)y(t+\tau)\,dt = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x^*(t)y(t+\tau)\,dt \qquad (\because x(t), y(t) \in \mathbb{R}) \\
&= \lim_{T\to\infty}\frac{1}{T}\int_{-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{\omega} X^*(\omega)e^{-j\omega t}\,d\omega\right)\left(\frac{1}{2\pi}\int_{\omega'} Y(\omega)e^{j\omega'(t+\tau)}\,d\omega'\right)dt \\
&= \lim_{T\to\infty}\frac{1}{2\pi T}\int_{\omega} X^*(\omega)e^{j\omega\tau}\int_{\omega'} Y(\omega')\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j(\omega'-\omega)t}\,dt\right)d\omega'\,d\omega \\
&= \lim_{T\to\infty}\frac{1}{2\pi T}\int_{\omega} X^*(\omega)e^{j\omega\tau}\int_{\omega'} Y(\omega')\delta(\omega'-\omega)\,d\omega'\,d\omega = \lim_{T\to\infty}\frac{1}{2\pi T}\int_{\omega} X^*(\omega)Y(\omega)e^{j\omega\tau}\,d\omega \\
&= \frac{1}{2\pi}\int_{\omega}\lim_{T\to\infty}\frac{X^*(\omega)Y(\omega)}{T}e^{j\omega\tau}\,d\omega = \frac{1}{2\pi}\int_{\omega} S_{xy}(\omega)e^{j\omega\tau}\,d\omega
\end{aligned}$$

Characteristics of Cross-spectrum

$$C_{xy}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)y(t+\tau)\,dt = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} y(t)x(t-\tau)\,dt = C_{yx}(-\tau)$$

$$\to C_{xy}(\tau) \in \mathbb{R} \qquad (\because x(t), y(t) \in \mathbb{R})$$

$$S_{xy}(\omega) = \mathcal{F}\left\{C_{xy}(\tau)\right\}$$

$$C_{xy}(-\tau) = C_{yx}(\tau) \tag{15}$$

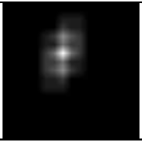
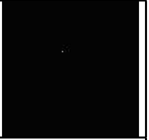
$$S_{xy}(-\omega) = S^*_{xy}(\omega) \tag{16}$$

$$S_{yx}(\omega) = S^*_{xy}(\omega) \tag{17}$$

$$S_{xy}(-\omega) = S_{yx}(\omega) \tag{18}$$

Evaluation of translation

$g(\boldsymbol{r}) = f(\boldsymbol{r} + \boldsymbol{\Delta})$, $\boldsymbol{\Delta} = (-8, 16)$ (Image size:128x128)

$f(\boldsymbol{r})$	$g(\boldsymbol{r})$	$C_{fg}(\boldsymbol{r})$ (w/o POC)	$\widehat{C}_{fg}(\boldsymbol{r})$ (w/ POC)
			
		$(-8, 16)$	$(-8.0, 16.0)$

$$\left(\begin{array}{l} C_{fg} = \mathcal{F}^{-1}\left\{\langle F^* G\rangle_t\right\} \\ \widehat{C}_{fg} = \mathcal{F}^{-1}\left\{\left\langle \widehat{F}^* \widehat{G}\right\rangle_t\right\} \\ \widehat{F} = \dfrac{F}{|F|},\ \widehat{G} = \dfrac{G}{|G|} \\ \widehat{F}^* \widehat{G} = e^{i(\phi_G - \phi_F)} \end{array} \right)$$

POC:Phase Only cross-Correlation

In the case of $g(\boldsymbol{r}) = f(\boldsymbol{r} + \boldsymbol{\Delta})$

$$G(\boldsymbol{k}) = e^{i\boldsymbol{k}\cdot\boldsymbol{\Delta}} F(\boldsymbol{k})$$

$$\widehat{F}^*(\boldsymbol{k})\widehat{G}(\boldsymbol{k}) = e^{i\boldsymbol{k}\cdot\boldsymbol{\Delta}}$$

($\phi_F(\boldsymbol{k})$ is canceled.)

$$\widehat{C}_{fg}(\boldsymbol{r}) = \mathcal{F}^{-1}\left\{\left\langle e^{i\boldsymbol{k}\cdot\boldsymbol{\Delta}}\right\rangle_r\right\}$$

$$= \lim_{L\to\infty} \frac{1}{L^2}\frac{1}{(2\pi)^2}\int e^{i\boldsymbol{k}\cdot(\boldsymbol{\Delta}-\boldsymbol{r})}\, d\boldsymbol{k}^2$$

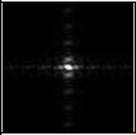
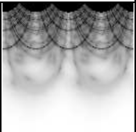
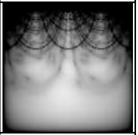
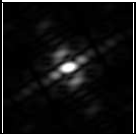
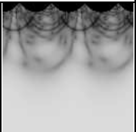
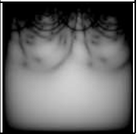
$$= \lim_{L\to\infty} \frac{1}{L^2}\delta(\boldsymbol{r} - \boldsymbol{\Delta}) \to \boxed{\text{Sharp peak}}$$

Evaluation of rotation and scaling

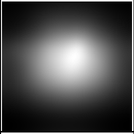
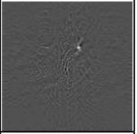
$$g(\boldsymbol{r}) = f(\alpha\boldsymbol{\Theta}\cdot\boldsymbol{r} + \boldsymbol{\Delta}),\quad \boldsymbol{\Delta} = (-8, 16),$$

$$\alpha = \frac{1}{2},\ \boldsymbol{\Theta} = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix},\ \theta_1 = 30\,\mathrm{deg}$$

(Image size:128x128)

(x, y)	Cartesian (k_x, k_y)	log-polar $(k_\theta, \log\lvert\boldsymbol{k}\rvert)$	Hanning-Win. $(k_\theta, \log\lvert\boldsymbol{k}\rvert)$
$f(\boldsymbol{r})$	$I_F(\boldsymbol{k}) = \lvert F(\boldsymbol{k})\rvert$		$\mathcal{H}\{I_F(\boldsymbol{k})\}$
			
$g(\boldsymbol{r})$	$I_G(\boldsymbol{k}) = \lvert G(\boldsymbol{k})\rvert$		$\mathcal{H}\{I_G(\boldsymbol{k})\}$
			
		(log scale [0.05, 500])	(log scale [0.05, 500])

$$C_{I_F I_G} = \mathcal{F}^{-1}\left\{(\mathcal{F}\{\mathcal{H}\{I_F\}\})^{*}\,\mathcal{F}\{\mathcal{H}\{I_G\}\}\right\}$$

	Cross-corr. of I_F and I_G $(k_\theta, \log\lvert\boldsymbol{k}\rvert)$ $C_{I_F I_G}(\boldsymbol{k})$	
	w/o POC	w/ POC
		
k_θ [deg]	19.6	29.4
$\log\lvert\boldsymbol{k}\rvert$	−0.343	−0.682
$\frac{1}{\lvert\boldsymbol{k}\rvert}$ [times]	0.710	0.505