# 1.3 Fast Fourier Transform (FFT)

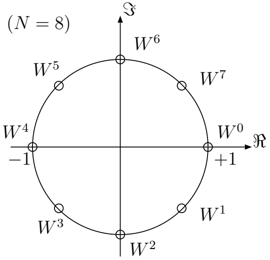
$$\hat{F}_m = \frac{1}{N}\sum_{n=0}^{N-1} f_n e^{-i\frac{2\pi}{N}mn} \quad (17)$$

$$F_m \triangleq \sum_{n=0}^{N-1} f_n \underbrace{e^{-i\frac{2\pi}{N}mn}}_{W^{mn}} \quad \left(\hat{F}_m = \frac{1}{N}F_m\right)$$

$$W = e^{-i\frac{2\pi}{N}}$$

$$F_m = \sum_{n=0}^{N-1} f_n W^{mn}$$

(FFT: Fast Fourier Transform)




In the case of $N = 2^p$ $(p \in \mathbb{N})$, $W^{mn \pm N/2} = -W^{mn}$ is satisfied.
By using this relation, the number of times of multiplications can be reduced.

Divide $n$ of $f_n$, $n \in \{0, \cdots, N-1\}$, into two groups which are the even group and the odd group.

$$f^e_{n'} = f_{2n'}, \quad f^o_{n'} = f_{2n'+1} \qquad (n' \in \{0, 1, ..., N/2-1\})$$

$$\left( \begin{aligned} F_m &= \sum_{n=0}^{N-1} f_n W^{mn} = \sum_{n'=0}^{N/2-1} f^e_{n'} W^{2mn'} + \sum_{n'=0}^{N/2-1} f^o_{n'} W^{m(2n'+1)} \\ &= \underbrace{\sum_{n=0}^{N/2-1} f^e_n W^{2mn}}_{\text{DFT of } f^e_n \equiv F^e_m (N/2\text{points})} + W^m \underbrace{\sum_{n=0}^{N/2-1} f^o_n W^{2mn}}_{\text{DFT of } f^o_n \equiv F^o_m (N/2\text{points})} = F^e_m + W^m F^o_m \end{aligned} \right)$$

Replace $(m \to N/2 + m)$

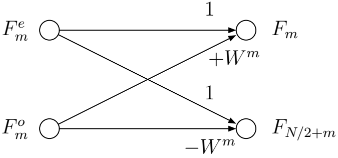
$$\left( \begin{aligned} F_{\frac{N}{2}+m} &= \sum_{n=0}^{N/2-1} f^e_n \overbrace{W^{2(N/2+m)n}}^{W^{2mn}} + \overbrace{W^{N/2+m}}^{-W^m} \sum_{n=0}^{N/2-1} f^o_n \overbrace{W^{2(N/2+m)n}}^{W^{2mn}} \\ &= \underbrace{\sum_{n=0}^{N/2-1} f^e_n W^{2mn}}_{\text{DFT of } f^e_n \equiv F^e_m (N/2\text{points})} - W^m \underbrace{\sum_{n=0}^{N/2-1} f^o_n W^{2mn}}_{\text{DFT of } f^o_n \equiv F^o_m (N/2\text{points})} = F^e_m - W^m F^o_m \end{aligned} \right)$$

# Butterfly operation

$$F_m^{\left\{ {e \atop o} \right\}} = \sum_{n=0}^{N/2-1} f_n^{\left\{ {e \atop o} \right\}} W^{2nm} \quad (22)$$

$$\begin{cases} F_m = F_m^e + W^m F_m^o \\ F_{N/2+m} = F_m^e - W^m F_m^o \end{cases} \quad (23)$$

$$(m \in \{0, \cdots, N/2-1\})$$

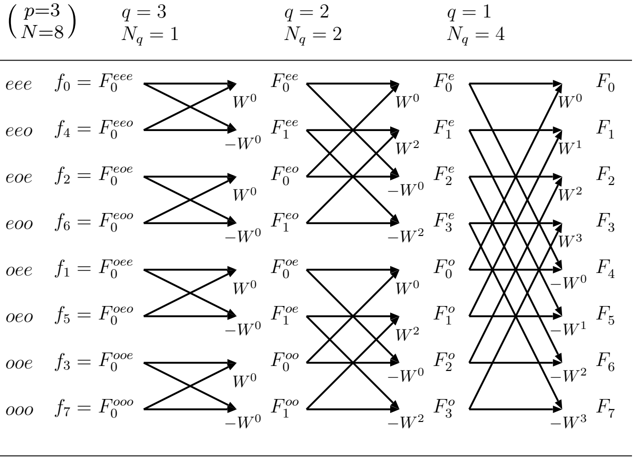


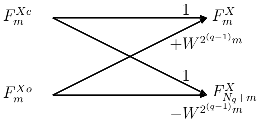
Butterfly operation

- If we know both $F_m^e$ and $F_m^o$, we can evaluate both $F_m$ and $F_{N/2+m}$.
- In order to evaluate $F_m$ and $F_{N/2+m}$, $N/2$ times of multiplications for each are needed. The sum of them are $N$ times.
- In order to obtain $F_m^e$ and $F_m^o$, $\cdots$.

Divide each of $F_m^e$ and $F_m^o$ into two groups. Furthermore, repeat dividing.

- 1st time $(N_1 = N/2)$ $F_{\left\{\substack{m \\ m+N_1}\right\}} = F_m^e \pm W^m F_m^o$
- 2nd time $(N_2 = N/2^2)$ $F^x_{\left\{\substack{m \\ m+N_2}\right\}} = F_m^{xe} \pm W^{2m} F_m^{xo} \quad (x \in \{e, o\})$
- 3rd time $(N_3 = N/2^3)$ $F^{x_1 x_2}_{\left\{\substack{m \\ m+N_3}\right\}} = F_m^{x_1 x_2 e} \pm W^{2^2 m} F_m^{x_1 x_2 o}$ $(x_0, x_1 \in \{e, o\})$
- $q$-th time $(N_q = N/2^q)$ $F^{\boldsymbol{X}}_{\left\{\substack{m \\ m+N_q}\right\}} = F_m^{\boldsymbol{X} e} \pm W^{2^{(q-1)} m} F_m^{\boldsymbol{X} o}$ $(\boldsymbol{X} = (x_0\ x_1\ \cdots x_{q-2}), x_i \in \{e, o\})$




$X = (x_0, ..., x_{p-q-1})$

$x_i \in \{e, o\}$

Times of multiplications:

$N$/stage $\times$ $p$ stage $= N \log_2 N$

# Bit-reversal scheme

**Relation between $F_m^{\boldsymbol{X}}$ and $f_n$ (e.g. $N=8, p=3$)**

| $F_m^{eo}$ ( Pick up the even group in the first. After that pick up the odd group. ) | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0 (0) | 1 (1) | 2 (2) | 3 (3) | 4 (4) | 5 (5) | 6 (6) | 7 (7) | pick up the even group ($e$) |
| 0 (0) | | 2 (1) | | 4 (2) | | 6 (3) | | pick up the even group ($o$) |
| | | 2 (0) | | | | 6 (1) | | $\rightarrow F_m^{eo}$ : FT of two points for $f_2$ and $f_6$. |
| $F_0^{eoo}$ ( Pick up groups in order of even, odd, and odd.) | | | | | | | | |
| | | | | | | 6 | | $\rightarrow F_0^{eoo}$ :FT of the single point for $f_6$. |

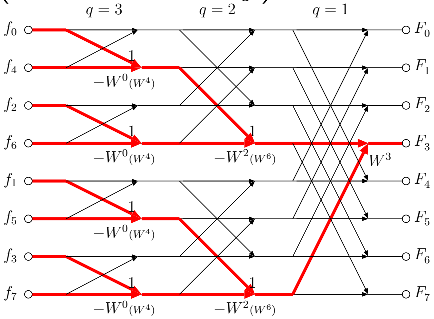
## Bit-reversal scheme

| (a) | parity | $eee$ | $eeo$ | $eoe$ | $eoo$ | $oee$ | $oeo$ | $ooe$ | $ooo$ |
|---|---|---|---|---|---|---|---|---|---|
| | | $e\rightarrow 0,\ o\rightarrow 1$ | | | | | | | |
| (b) | binary | 000 | 001 | 010 | 011 | 100 | 101 | 110 | 111 |
| | (decimal) | (0) | (1) | (2) | (3) | (4) | (5) | (6) | (7) |
| (c) | reversal | 000 | 100 | 010 | 110 | 001 | 101 | 011 | 111 |
| | (decimal) | (0) | (4) | (2) | (6) | (1) | (5) | (3) | (7) |

We can obtain the list of $n$ for $f_n$ by using bit reverse.

# Example of FFT operation

(Path to evaluate $F_3$.)




| Product along path ($W^8=1, -1=W^4$) | | | | | check | | |
|---|---|---|---|---|---|---|---|
| beg. | St. 1 | St. 2 | St. 3 | Product ○ | $n$ | $3n$ | $3n\%8$ ○ |
| $f_0$ | 1 | 1 | 1 | $1=W^0$ | 0 | 0 | 0 |
| $f_4$ | $-W^0$ | 1 | 1 | $-W^0=W^4$ | 4 | 12 | 4 |
| $f_2$ | 1 | $-W^2$ | 1 | $-W^2=W^6$ | 2 | 6 | 6 |
| $f_6$ | $-W^0$ | $-W^2$ | 1 | $W^2$ | 6 | 18 | 2 |
| $f_1$ | 1 | 1 | $W^3$ | $W^3$ | 1 | 3 | 3 |
| $f_5$ | $-W^0$ | 1 | $W^3$ | $-W^3=W^7$ | 5 | 15 | 7 |
| $f_3$ | 1 | $-W^2$ | $W^3$ | $-W^5=W^1$ | 3 | 9 | 1 |
| $f_7$ | $-W^0$ | $-W^2$ | $W^3$ | $W^5$ | 7 | 21 | 5 |

We can confirm that the columns with "○" are same.

$$F_m = \sum_{n=0}^{N-1} f_n W^{nm}$$

# To apply FFT

- The FFT can be applied only when $N = 2^p$.
- In the case of $N \neq 2^p$ $(2^{p-1} < N < 2^p)$:
  - Remove data with less information around ends so that $N' = 2^{p-1}$.
  - Add data $f_n = f^\dagger$ for $n \in N, ..., 2^p$ so that $N' = 2^p$. (padding)
    $\left(\text{Padding data } f^\dagger\text{: } \bar{f}\text{(ave.), } 0\text{, or } \frac{f_0 + f_{N-1}}{2}\right)$
  - A window function after adding or removing is multiplied, if necessary.

# Summary of FFT

- When $N = 2^p$, the times of multiplications can be reduced by using butterfly operations.
- Comparison of the times of multiplications

$$N_{\mathsf{Mul}}(\mathsf{FFT}) = N \log_2 N \qquad \rightarrow \qquad N_{\mathsf{Mul}}(\mathsf{FFT}) \ll N_{\mathsf{Mul}}(\mathsf{DFT})$$
$$N_{\mathsf{Mul}}(\mathsf{DFT}) = N^2$$

  FFT is more effective with increasing $N$.

  e.g.

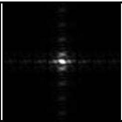
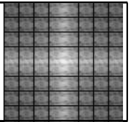
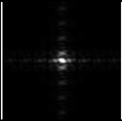
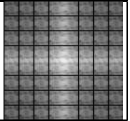
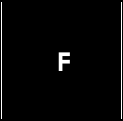
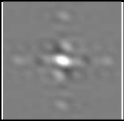
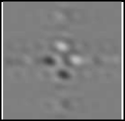
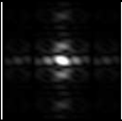
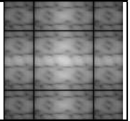
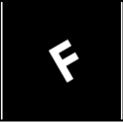
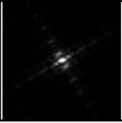
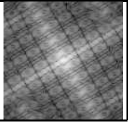
| $N$ | 32 | 1024 | 32768 |
|---|---|---|---|
| DFT | $\sim 1000$ | $\sim 10^6$ | $\sim 10^9$ |
| FFT | $160$ | $\sim 10^4$ | $\sim 5 \times 10^5$ |

- In the case of a two-dimensional image ($N_x \times N_y$), FFT can be applied only for the most inner loop.

$$N_{\mathsf{Mul}}(\mathsf{FFT}) = N_x N_y (\log_2 N_x + \log_2 N_y)$$
$$N_{\mathsf{Mul}}(\mathsf{DFT}) = N_x N_y (N_x + N_y)$$

# 1.4 Characteristices of Fourier Transform

Examples of Fourier Transform (Origin : center)

| Ope. | $f(\boldsymbol{r}) \in \mathbb{R}$ | $\Re\{F(\boldsymbol{k})\}$ | $\Im\{F(\boldsymbol{k})\}$ | $\lvert F(\boldsymbol{k})\rvert$ | $\log \lvert F(\boldsymbol{k})\rvert$ |
|---|---|---|---|---|---|
| Orig. |  |  |  |  |  |
| Shift |  |  |  |  |  |
| Scale down |  |  |  |  |  |
| Rotation |  |  |  |  |  |

# Symmetry

$$f(\boldsymbol{r}) = a(\boldsymbol{r}) + ib(\boldsymbol{r}) \quad (a, b \in \mathbb{R},\ f \in \mathbb{C})$$

$$F(\boldsymbol{k}) = A(\boldsymbol{k}) + iB(\boldsymbol{k}) \quad (A, B, F \in \mathbb{C})$$
$$= A + iB \qquad (\text{omit } (\boldsymbol{k}))$$

$$F(\boldsymbol{k}) \triangleq \int f(\boldsymbol{r}) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\, d\boldsymbol{r}$$

$$\left( \begin{array}{l} \begin{Bmatrix} A(\boldsymbol{k}) \\ B(\boldsymbol{k}) \end{Bmatrix} = \int \begin{Bmatrix} a(\boldsymbol{r}) \\ b(\boldsymbol{r}) \end{Bmatrix} e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\, d\boldsymbol{r} \\ \begin{Bmatrix} A(-\boldsymbol{k}) \\ B(-\boldsymbol{k}) \end{Bmatrix} = \int \begin{Bmatrix} a(\boldsymbol{r}) \\ b(\boldsymbol{r}) \end{Bmatrix} e^{+i\boldsymbol{k}\cdot\boldsymbol{r}}\, d\boldsymbol{r} \end{array} \right)$$

$$\begin{Bmatrix} A(-\boldsymbol{k}) \\ B(-\boldsymbol{k}) \end{Bmatrix} = \begin{Bmatrix} A^*(\boldsymbol{k}) \\ B^*(\boldsymbol{k}) \end{Bmatrix} = \begin{Bmatrix} A^* \\ B^* \end{Bmatrix}$$

$$F^*(\boldsymbol{k}) = A^* - iB^*$$

$$F(-\boldsymbol{k}) = A^* + iB^* \neq F^*(\boldsymbol{k})$$

$$|F(\boldsymbol{k})|^2 = (A + iB)^*(A + iB)$$
$$= (|A|^2 + |B|^2) + i(A^*B - AB^*)$$

$$|F(-\boldsymbol{k})|^2 = (A^* + iB^*)^*(A^* + iB^*)$$
$$= (|A|^2 + |B|^2) - i(A^*B - AB^*)$$
$$\neq |F(\boldsymbol{k})|^2$$

| | $\Re\{F(\boldsymbol{k})\}$ | $\Im\{F(\boldsymbol{k})\}$ | $\lvert F(\boldsymbol{k})\rvert$ |
|---|---|---|---|
| $f(\boldsymbol{r}) = a(\boldsymbol{r})$(Real) | sym. | anti-sym. | sym. |
| $f(\boldsymbol{r}) = ib(\boldsymbol{r})$(Pure imag.) | anti-sym. | sym. | sym. |
| $f(\boldsymbol{r}) = a(\boldsymbol{r}) + ib(\boldsymbol{r})$(Complex) | non-sym. | non-sym. | non-sym. |

# Coordinate transformation

| | $\Re\{F(\boldsymbol{k})\}$ | $\Im\{F(\boldsymbol{k})\}$ | $\lvert F(\boldsymbol{k})\rvert$ |
|---|---|---|---|
| Translation | | | Identical |
| Scale down | Scale up | Scale up | Scale up |
| Rotation | Rotation | Rotation | Rotation |

## Translation

$$f_1(\boldsymbol{r}) = f_0(\boldsymbol{r}+\boldsymbol{\Delta})$$

$$F_1(\boldsymbol{k}) = e^{+i\boldsymbol{k}\cdot\boldsymbol{\Delta}} F_0(\boldsymbol{k})$$

$$\left(\begin{array}{l} \quad(\boldsymbol{r}' = \boldsymbol{r}+\boldsymbol{\Delta}) \\ F_1(\boldsymbol{k}) = \int f_1(\boldsymbol{r}) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\,d\boldsymbol{r} \\ \quad = \int f_0(\boldsymbol{r}+\boldsymbol{\Delta}) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\,d\boldsymbol{r} \\ \quad = \int f_0(\boldsymbol{r}') e^{-i\boldsymbol{k}\cdot(\boldsymbol{r}'-\boldsymbol{\Delta})}\,d\boldsymbol{r}' \\ \quad = e^{+i\boldsymbol{k}\cdot\boldsymbol{\Delta}} \\ \qquad \cdot \int f_0(\boldsymbol{r}') e^{-i\boldsymbol{k}\cdot\boldsymbol{r}'}\,d\boldsymbol{r}' \end{array}\right)$$

## Scaling

$$f_1(\boldsymbol{r}) = f_0(\alpha\boldsymbol{r})$$

$$F_1(\boldsymbol{k}) = \frac{1}{\alpha} F_0\left(\frac{\boldsymbol{k}}{\alpha}\right)$$

$$\left(\begin{array}{l} \quad(\boldsymbol{r}' = \alpha\boldsymbol{r}) \\ F_1(\boldsymbol{k}) = \int f_1(\boldsymbol{r}) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\,d\boldsymbol{r} \\ \quad = \int f_0(\alpha\boldsymbol{r}) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\,d\boldsymbol{r} \\ \quad = \int f_0(\boldsymbol{r}') e^{-i\boldsymbol{k}\cdot\frac{\boldsymbol{r}'}{\alpha}}\,\frac{d\boldsymbol{r}'}{\alpha} \\ \quad = \frac{1}{\alpha}\int f_0(\boldsymbol{r}') e^{-i\left(\frac{\boldsymbol{k}}{\alpha}\right)\cdot\boldsymbol{r}'}\,d\boldsymbol{r}' \end{array}\right)$$

## Rotation

$$f_1(\boldsymbol{r}) = f_0(\boldsymbol{\Theta}\cdot\boldsymbol{r})$$

$$F_1(\boldsymbol{k}) = F_0\left(\boldsymbol{\Theta}\cdot\boldsymbol{k}\right)$$

$$\left(\begin{array}{l} \quad(\boldsymbol{r}' = \boldsymbol{\Theta}\cdot\boldsymbol{r}) \\ F_1(\boldsymbol{k}) = \int f_1(\boldsymbol{r}) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\,d\boldsymbol{r} \\ \quad = \int f_0(\boldsymbol{\Theta}\boldsymbol{r}) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\,d\boldsymbol{r} \\ \quad = \int f_0(\boldsymbol{r}') e^{-i\boldsymbol{k}\cdot\boldsymbol{\Theta}^{-1}\cdot\boldsymbol{r}'}\,\frac{d\boldsymbol{r}'}{|\boldsymbol{\Theta}|} \\ \qquad \left(\begin{array}{l} \boldsymbol{k}\cdot\boldsymbol{\Theta}^{-1} = \boldsymbol{\Theta}\cdot\boldsymbol{k}, \\ |\boldsymbol{\Theta}| = 1 \end{array}\right) \\ \quad = \int f_0(\boldsymbol{r}') e^{-i(\boldsymbol{\Theta}\cdot\boldsymbol{k})\cdot\boldsymbol{r}'}\,d\boldsymbol{r}' \end{array}\right)$$