1.2 Discrete Fourier Transform (DFT)

$$f'(x) = \sum_{m=-\infty}^{\infty} F'_{k_m} e^{ik_m x} \quad (4)$$

$$F'_{k_m} = \frac{1}{L}\int_0^L f'(x) e^{-ik_m x}\, dx \quad (5)$$

$$k_m = m\frac{2\pi}{L} \quad (15)$$

(In exact, "Discrete Fourier Series Expansion".)

Consider the case when $f(x)$ is sampled with even intervals.

$$(n = 0, 1, \cdots, N-1)$$

$$f_n \equiv f'(x_n), \quad x_n \equiv n\Delta x$$

(Represent integral by summation:

$$\int_0^L \cdots\, dx \simeq \sum_{n=0}^{N-1} \cdots\, \Delta x$$

$$(L = N\Delta x)$$

$$\to F'_{k_m} \simeq \frac{1}{L}\sum_{n=0}^{N-1} f_n e^{-ik_m x_n} \Delta x$$
)

$$f_n = \sum_{m=0}^{N-1} F_m e^{+i\frac{2\pi nm}{N}} \quad (16)$$

$$F_m = \frac{1}{N}\sum_{n=0}^{N-1} f_n e^{-i\frac{2\pi nm}{N}} \quad (17)$$

Periodicity of F_m and f_n

$$F_{m'} = \frac{1}{N}\sum_{n=0}^{N-1} f_n e^{-i\frac{2\pi n m'}{N}} \quad (17)$$

Replacing $m' = m + N$

$$\begin{aligned} F_{m+N} &= \frac{1}{N}\sum_{n=0}^{N-1} f_n\, e^{-i\frac{2\pi n(m+N)}{N}} \\ &= \frac{1}{N}\sum_{n=0}^{N-1} f_n\, e^{-i\frac{2\pi nm}{N}} \underbrace{e^{-i2\pi n}}_{=1} = F_m \end{aligned}$$

F_m has a periodicity.

$$F_{m\pm N} = F_m \quad (18)$$

$$f_{n'} = \sum_{m=0}^{N-1} F_m e^{+i\frac{2\pi n' m}{N}} \quad (16)$$

F_m also has a periodicity.

$$f_{n\pm N} = f_n \quad (19)$$

We can choose any set of N points for m in F_m or n in f_n, if the point is not a periodic points of others.
However, we must consider the sampling theorem for interpolations.

Sampling Theorem

Signal consist of single sinusoidal function:

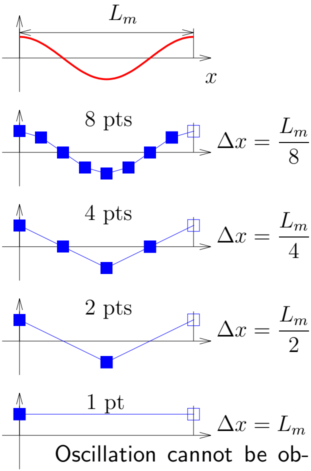
$$f_m(x) = F_m e^{ik_m x} \quad (k_m = \frac{2\pi}{L_m})$$

In order to observe oscillation with a period L_m, two points are needed within L_m.

Sampling Theorem

$$\Delta x \leq \frac{L_m}{2}$$

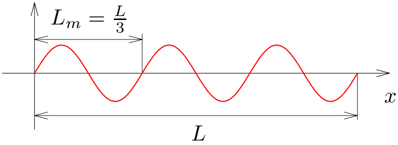
$$k_m \leq \frac{\pi}{\Delta x} \quad \text{(Nyquist Freq.)} \tag{20}$$



Oscillation cannot be observed, with $\Delta x = L_m$.

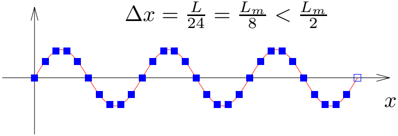
Sampling Theorem and Aliasing

Case of $m = 3$:



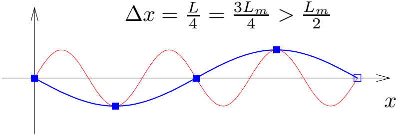
Fine sampling $(N = 24)$

$\Delta x = \frac{L}{24} = \frac{L_m}{8} < \frac{L_m}{2}$



Coarse sampling $(N = 4)$

$\Delta x = \frac{L}{4} = \frac{3L_m}{4} > \frac{L_m}{2}$



$\Delta x > \dfrac{L}{2}$ → Different sinusoidal function is observed. → Aliasing

$$\left(\begin{array}{l} \text{Freq. of true signal:} \\ \quad k_m = m\dfrac{2\pi}{L} \\ \text{Freq. of spurious signal:} \\ \quad k'_m = (m - N)\dfrac{2\pi}{L} = k_{m-N} \end{array}\right)$$

If the sampling interval is Δx, the signal with $k_m > \dfrac{\pi}{\Delta x}$ cannot be observed.

→Sampling Theorem

In this condition, the aliased signal with the following frequency is observed.

$$\frac{-\pi}{\Delta x} \le k_{m'} = k_{m-N} \le \frac{\pi}{\Delta x}$$

$$(|m - N| \le N/2)$$

Relation of domains and intervals between real and Fourier spaces

$$\begin{cases} L = x_{\max} - x_{\min} = N\Delta x & (f_n = f_{n\pm N}) \\ K = k_{\max} - k_{\min} = N\Delta k & (F_m = F_{m\pm N}) \end{cases}$$

- Sampling Theorem: $|k| < \dfrac{\pi}{\Delta x}$
- Symmetry of k $(k_{\min} = -k_{\max})$

$$\left(k_{\max} = -k_{\min} = \frac{\pi}{\Delta x}\right)$$

$$K = k_{\max} - k_{\min} = \frac{2\pi}{\Delta x}$$

$$\Delta k = \frac{2\pi}{x_{\max} - x_{\min}} = \frac{2\pi}{L}$$

(Real space)		(Fourier space)
Δx	Reciprocal $\Longleftrightarrow$ (Prod.$=2\pi$)	K
$\times N \Downarrow$		$\Uparrow \times N$
L	Reciprocal $\Longleftrightarrow$ (Prod.$=2\pi$)	Δk

Summary of DFT

FT of signal with discrete sampling (num. of samp.$=N$)

$$f_n = \sum_{\substack{m \\ (x_n = n\Delta x)}} F_m e^{ik_m x_n} \quad \underset{\text{INV}}{\overset{\text{FWD}}{\rightleftarrows}} \quad F_m = \frac{1}{N} \sum_{\substack{n \\ (k_m = m\Delta k)}} f_n e^{-ik_m x_n}$$

- Suffixes n and m of f_n and F_m have periodicity with the period N.
- In the computation of f_n and F_m, arbitrary set of m and n has same result because of this periodicity. (eg. $n, m = \{0, \cdots, N-1\}$, or $n, m = \{\lfloor -N/2 \rfloor, \cdots, \lfloor N/2 - 1 \rfloor\}$)
- However, if m of F_m is $|m| > N/2$, m should be shifted into $|m| \leq N/2$ to satisfy sampling theorem.
- Interpolation

  Once F_m is obtained, we can evaluate $f(x|x \neq n\Delta x)$ by inverse transform. In this case, m must satisfy the sampling theorem. Otherwise, the interpolated function shows an aliasing function.

Techniques to compute DFT

$$F_m = \frac{1}{N}\sum_{n=0}^{N-1} f_n e^{-i\frac{2\pi mn}{N}} \qquad (17)$$

- In a simple computation, most of computational cost is consumed to evaluate exponential function $e^{-i\frac{2\pi mn}{N}}$.
- To evaluate all of F_m, N^2 times evaluations are needed.
- Argument of exponential function

$$\frac{mn}{N} = \underbrace{\left\lfloor \frac{mn}{N} \right\rfloor}_{\text{Integer}} + \frac{1}{N}\underbrace{(mn)\%N}_{\text{Fraction}}$$

$$mn\%N \in \{0, 1, \cdots, N-1\}$$

- Since $e^{i2l\pi} = 1$ for $l \in \mathbb{Z}$,

$$e^{-i\frac{2\pi mn}{N}} = e^{-i\frac{2\pi(mn)\%N}{N}}.$$

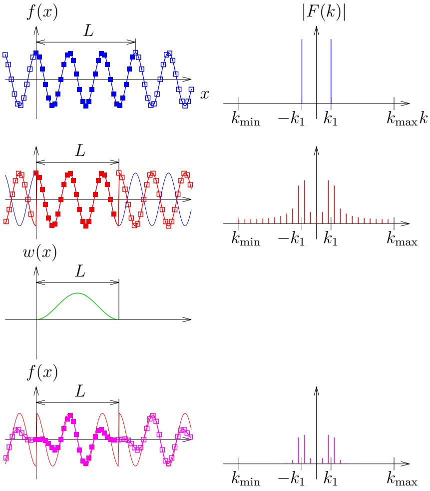
If we evaluate $W^p = e^{-\frac{i2\pi}{N}p}$ for $p \in \{0, 1, \cdots, N\}$ at the first, the number of times for exponential evaluations is only N.

(When we use this table, the time to compute multiplication governs the computational time.
The scheme to reduce the num. of times for multiplication is called FFT.)

Window function

Sinusoidal signal: $f_{x_n} = \cos k_1 x_n$

- Case of $f_{x_0} = f_{x_N}$
  $|F_{k_m}| = \frac{1}{2}(\delta_{m,n} + \delta_{-m,n})$
- Case of $f_{x_0} \neq f_{x_N}$
  $|F(k)|$ spreads around k_1.
  - Reason:
    Assumed periodicity.
  - Solution: Multiply window function $w(x)$ so that the ends becomes continuous.
    $$f'(x) = w(x) f(x)$$
  - example of the window function:
    $$w(x) = \frac{1}{2}\left(1 + \cos\frac{2\pi(x - x_c)}{L}\right)$$
    $$(x \in [x_c - L/2, x_c + L/2])$$



Two-dimensional DFT

$$F_{m_x,m_y} = \frac{1}{N_x N_y} \sum_{n_x=0}^{N_x-1} \sum_{n_y=0}^{N_y-1} f_{n_x,n_y} e^{-i\left(\frac{2\pi m_x n_x}{N_x} + \frac{2\pi m_y n_y}{N_y}\right)} \tag{21}$$

To evaluate all F_{m_x,m_y} $(N_x N_y)^2$ times multiplication are needed. (4 loops)

(Exchange the order of operators)

$$F_{m_x,m_y} = \frac{1}{N_x} \sum_{n_x=0}^{N_x-1} \underbrace{\left\{ \frac{1}{N_y} \sum_{n_y=0}^{N_y-1} f_{n_x,n_y} e^{-i\frac{2\pi m_y n_y}{N_y}} \right\}}_{=G_{n_x,m_y}} e^{-i\frac{2\pi m_x n_x}{N_x}}$$

$$G_{n_x,m_y} = \frac{1}{N_y} \sum_{n_y=0}^{N_y-1} f_{n_x,n_y} e^{-i\frac{2\pi m_y n_y}{N_y}}, \qquad F_{m_x,m_y} = \frac{1}{N_x} \sum_{n_x=0}^{N_x-1} G_{n_x,m_y} e^{-i\frac{2\pi m_x n_x}{N_x}}$$

$\begin{Bmatrix} G_{n_x,m_y} \\ F_{m_x,m_y} \end{Bmatrix}$ is the FT of $\begin{Bmatrix} f_{n_x,n_y} \\ G_{n_x,m_y} \end{Bmatrix}$ with $\begin{Bmatrix} m_y \\ m_x \end{Bmatrix}$. The num. of multi. for all is $\begin{Bmatrix} N_x N_y^2 \\ N_x^2 N_y \end{Bmatrix}$.
Total num. of operation is reduced to $N_x N_y^2 + N_x^2 N_y$ times. (3 loops for each step. 2 steps.)