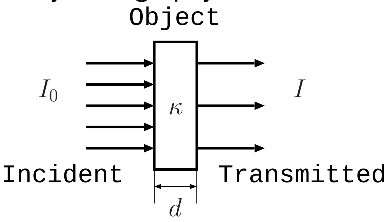
9. Computed Tomography (CT)
9.1 Absorption of X-ray

- X-ray radiography



$$I(y) = I_0 e^{-\kappa(y) d}$$

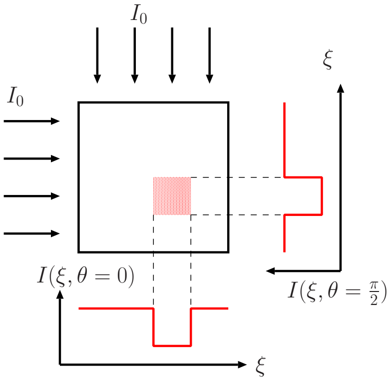
- κ : attenuation coefficient
  In the case of X-ray,
  it depends on the atomic number.
  (Heavy atom $\rightarrow$ large κ.)

- If κ is depth dependent,

$$\kappa d \rightarrow \int_0^d \kappa(x,y)\,dx$$
$$\rightarrow \int_{-\infty}^{\infty} \kappa(x,y)\,dx$$
$$(\kappa(x,y) = 0 \quad \text{Not Object})$$

- Only the integral of κ along optical path can be obtained from X-ray radiography.

9.2 Projection from several directions

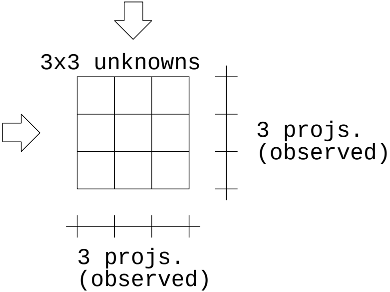


The distribution of κ including depth distribution, which is called tomography, can be obtained from several projected data with different directions.

$\Downarrow$

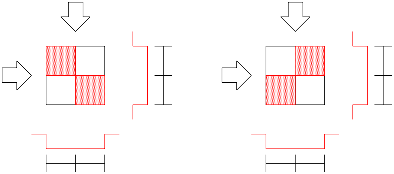
Computed Tomography

Number of projections and Num. of internal nodes



Num. of unknowns $>$ Num. of Obs.

$\rightarrow$ Cannot solve.



Num. of unknowns $=$ Num. of Obs.

$\rightarrow$ Cannot distinguish.

To obtain more projection, other projections with different directions are needed.

$$\theta \in [0, \pi]$$

9.3 Schematic of forward and backward-projection
Forward projection (measuring process)

Forward projection

$\equiv$Integral along beam path

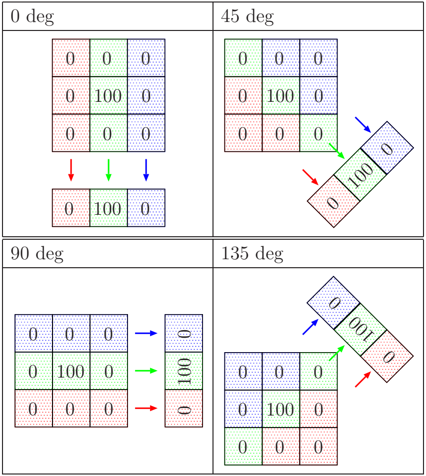
$$p(\xi, \theta) = \int_\eta \kappa(x, y)\, d\eta_\theta$$

$$(x \equiv x(\xi, \theta),\ \ y \equiv y(\xi, \theta))$$

$\rightarrow$ Accumulate along path

0	0	0
0	100	0
0	0	0

$\Longrightarrow$



Backward projection (1) Simple backprojection

1. Map averaged value of the projection data along path for each angle.

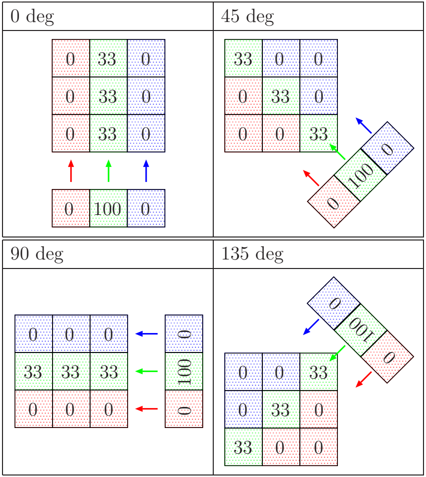
2. Take an average of mapped data for each pixel.

8	8	8
8	33	8
8	8	8

Blurrier than original.

Original

0	0	0
0	100	0
0	0	0



Backward projection (2) Filtered backward projection

In the simple backward projection, the reconstracted result is blurred. To reduce the blurring, edge enhancement filter is applied to the projection data.

Edge enhancement filter

$$\begin{aligned} g_n &= f_n - k f_n'' \\ &= f_n - k(f_{n-1} - 2f_n + f_{n+1}) \\ &= -k f_{n-1} + (2k+1) f_n - k f_{n+1} \end{aligned}$$

$(k = 1)$

$$g_n = \sum_{i=-1}^{+1} w_i f_{n-i}$$

$$(w_{-1}, w_0, w_{+1}) = (-1, +3, -1)$$

Filtered backward projection

1. Edge enhancement of projection data: p'

   (e.g.) $p'_j = \sum_{i=-1}^{+1} w_i p_{j-i}$

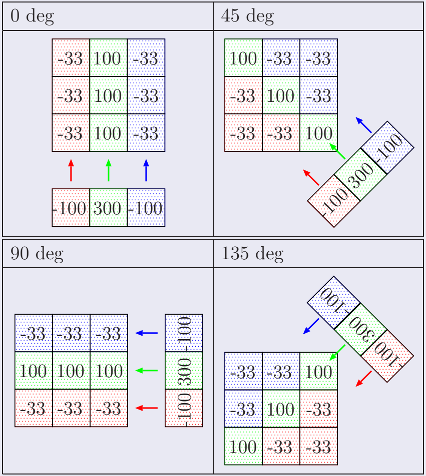
   $(w_{-1}, w_0, w_{+1}) = (-1, 3, -1)$

p	0	100	0
p'	-100	300	-100

2. Apply simple backward projection using p'.

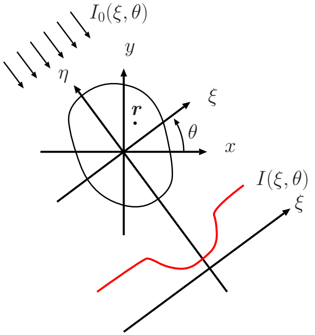
0	0	0
0	100	0
0	0	0

$\Longleftarrow$



In this example, the reconstructed field is identical to original.

9.4 Radon Transform
Coordinate system



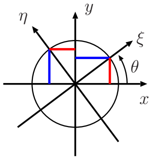
Expression of point $\boldsymbol{r}$

$$\boldsymbol{r} = x\,\boldsymbol{e}_x + y\,\boldsymbol{e}_y$$
$$= \xi\,\boldsymbol{e}_{\boldsymbol{\xi}} + \eta\,\boldsymbol{e}_{\boldsymbol{\eta}}$$

Inner product of $\boldsymbol{e}_x$:

$$x\,\boldsymbol{e}_x\cdot\boldsymbol{e}_x + y\,\boldsymbol{e}_x\cdot\boldsymbol{e}_y$$
$$= \xi\,\boldsymbol{e}_x\cdot\boldsymbol{e}_{\boldsymbol{\xi}} + \eta\,\boldsymbol{e}_x\cdot\boldsymbol{e}_{\boldsymbol{\eta}}$$

$$x = \xi\cos\theta - \eta\sin\theta$$



$$\begin{cases} x = +\xi\cos\theta - \eta\sin\theta \\ y = +\xi\sin\theta + \eta\cos\theta \end{cases}$$

$$\begin{cases} \xi = +x\cos\theta + y\sin\theta \\ \eta = -x\sin\theta + y\cos\theta \end{cases}$$

Radon Transform

- Projected data (known)

$$I(\xi,\theta) = I_0\, e^{-\int_L \kappa(\boldsymbol{r})\, dl}$$
$$L \in \{\boldsymbol{r}(\xi,\eta;\theta) \,|\, \xi = \xi'(\text{const})\}$$

- Sinogram (known)

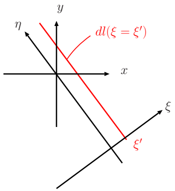
$$p(\xi,\theta) \equiv -\log \frac{I(\xi,\theta)}{I_0}$$

- Radon Transform
  Integral over straight line in 2-D space.

$$p(\xi,\theta) = \int_L \kappa(\boldsymbol{r}(\xi,\theta))\, dl$$

- Extend from line integral to 2-D area integral

$$\xi' = x\cos\theta + y\sin\theta$$



$$\int_L [\cdots]\, dl = \int_{\xi=\xi'} [\cdots]\, d\eta$$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [\cdots]\delta(\xi - \xi')\, d\xi'\, d\eta$$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [\cdots]\delta(\xi - \xi'(x,y))\, dx\, dy$$
$$= \iint [\cdots]\delta(\xi - (x\cos\theta + y\sin\theta))\, dx\, dy$$

9.5 Projection slice theorem

- Projection (Sinogram)

$$p(\xi, \theta) = \iint \kappa(x, y)\delta(\xi - (x\cos\theta + y\sin\theta))\, dx\, dy \qquad (1)$$

- FT with respect to ξ

$$\begin{aligned} P(k_\xi, \theta) &= \iiint \kappa(x, y)\delta(\xi - (x\cos\theta + y\sin\theta))e^{-jk_\xi \xi}\, dx\, dy\, d\xi \\ &= \iint \kappa(x, y)e^{-jk_\xi(x\cos\theta + y\sin\theta)}\, dx\, dy \qquad (2) \end{aligned}$$

2-D Fourier Transform in polar coordinate system

- Forward transform

$$F(k_x, k_y) = \iint f(x, y) e^{-j(k_x x + k_y y)}\, dx\, dy \qquad (k_x = k\cos\theta, \quad k_y = k\sin\theta)$$

$$= \iint f(x, y) e^{-jk(x\cos\theta + y\sin\theta)}\, dx\, dy \equiv F'(k, \theta) \tag{3}$$

- Inverse transform

$$f(x, y) = \frac{1}{4\pi^2} \iint F(k_x, k_y) e^{+j(k_x x + k_y y)}\, dk_x\, dk_y \qquad \left(\iint dk_x\, dk_y = \int_0^\infty \int_0^{2\pi} k\, d\theta\, dk\right)$$

$$= \frac{1}{4\pi^2} \int_0^\infty \int_0^{2\pi} F'(k, \theta) e^{+jk(x\cos\theta + y\sin\theta)}\, k\ d\theta\, dk \tag{4}$$

Eq. (2) and Eq. (3) are same.

Projection Slice Theorem

$P(k_\xi, \theta)$ is expressed by Fourier transform of $\kappa(x, y)$ in polar coordinate system.

Since $P(k_\xi, \theta)$ is known, $\kappa(x, y)$ is obtained by the inverse FT using (4).

$$\kappa(x, y) = \frac{1}{4\pi^2} \int_0^\infty \int_0^{2\pi} P(k_\xi, \theta) e^{+jk_\xi(x \cos\theta + y \sin\theta)} \, k_\xi \; d\theta \, dk_\xi \qquad (5)$$

Projection from opposite direction

$$\begin{aligned}\kappa(x,y) &= \frac{1}{4\pi^2}\int_0^{\infty}\int_0^{2\pi} P(k_\xi,\theta)e^{+jk_\xi(x\cos\theta+y\sin\theta)}\,k_\xi\;d\theta\,dk_\xi \\ &= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_0^{\pi} P(k_\xi,\theta)e^{+jk_\xi(x\cos\theta+y\sin\theta)}\,|k_\xi|\;d\theta\,dk_\xi\end{aligned}$$

$$\int_0^{\infty}\int_0^{2\pi} k_\xi\,d\theta\,dk_\xi = \int_0^{\infty}\int_0^{\pi} k_\xi\,d\theta\,dk_\xi + \int_0^{\infty}\int_\pi^{2\pi} k_\xi\,d\theta\,dk_\xi$$

Projection from opposite direction

$$\left(\begin{array}{c}\theta\to\theta\pm\pi\\ \xi\to-\xi\end{array}\right)\quad\left(\begin{array}{l}p(\xi,\theta\pm\pi)=p(-\xi,\theta)\\ P(k_\xi,\theta\pm\pi)=P(-k_\xi,\theta)\\ e^{+jk_\xi(x\cos(\theta\pm\pi)+y\sin(\theta\pm\pi))}=e^{-jk_\xi(x\cos\theta+y\sin\theta)}\end{array}\right)$$

$$\begin{aligned}\text{2nd Term} &= \int_0^{\infty}\int_\pi^{2\pi} P(k_\xi,\theta)e^{+jk_\xi(x\cos\theta+y\sin\theta)}k_\xi\,d\theta\,dk_\xi \qquad (\theta'=\theta-\pi)\\ &= \int_0^{\infty}\int_0^{\pi} P(-k_\xi,\theta')e^{-jk_\xi(x\cos\theta'+y\sin\theta')}k_\xi\,d\theta'\,dk_\xi \qquad (k'_\xi=-k_\xi)\\ &= \int_0^{-\infty}\int_0^{\pi} P(k'_\xi,\theta')e^{+jk'_\xi(x\cos\theta'+y\sin\theta')}(-k'_\xi)\,d\theta'\,(-dk'_\xi)\\ &= \int_{-\infty}^{0}\int_0^{\pi} P(k'_\xi,\theta')e^{+jk'_\xi(x\cos\theta'+y\sin\theta')}|k'_\xi|\,d\theta'\,dk'_\xi\end{aligned}$$

9.6 Reconstruction by using Fourier transform

$$\kappa(x,y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_0^{\pi} P(k_\xi,\theta) e^{+jk_\xi \overbrace{(x\cos\theta + y\sin\theta)}^{\xi}} |k_\xi| \, d\theta \, dk_\xi$$

$$= \frac{1}{2\pi} \int_0^{\pi} \underbrace{\frac{1}{2\pi} \int_{-\infty}^{\infty} [P(k_\xi,\theta)|k_\xi|] \, e^{+jk_\xi \xi} \, dk_\xi}_{=\mathcal{F}_{k_\xi}^{-1}\{P((k_\xi,\theta)|k_\xi|\} \equiv q(\xi,\theta)} \, d\theta = \frac{1}{2} \underbrace{\frac{1}{\pi} \int_0^{\pi} q(\xi(x,y),\theta) \, d\theta}_{\text{Average with } \theta.}$$

$$q(\xi(x,y),\theta) = \int_{-\infty}^{\infty} [P(k_\xi,\theta)|k_\xi|] \, e^{+jk_\xi (x\cos\theta + y\sin\theta)} \, dk_\xi$$

$$\kappa(x,y) = \frac{1}{2} \left\langle \mathcal{F}_{k_\xi}^{-1} \left\{ \mathcal{F}_\xi \{p(\xi,\theta)\}_\xi \, H(k_\xi) \right\}_{k_\xi} \right\rangle_\theta$$

($H(k_\xi) = |k_\xi|$ in the case of Ramp function)

1. Sinogram :
$$p(\xi,\theta)$$
2. FWD FT with ξ :
$$P(k_\xi,\theta) = \int_{-\infty}^{\infty} p(\xi,\theta)e^{-jk_\xi \xi}\, d\xi$$
3. Filtering (Weight with $|k_\xi|$) :
$$|k_\xi|P(k_\xi,\theta)$$
4. INV FT with k_ξ :
$$q(\xi,\theta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} P(k_\xi,\theta)|k_\xi|e^{+jk_\xi \xi}\, dk_\xi$$
5. Backward projection (Coordinate transform and integrate with θ) :
$$\kappa(x,y) = \frac{1}{2\pi}\int_{0}^{\pi} q(x\cos\theta + y\sin\theta,\theta)\, d\theta$$

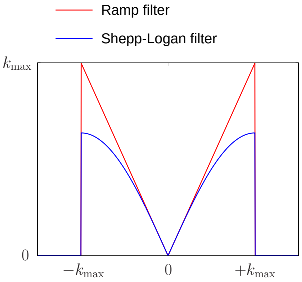
Two FT (FWD and INV) are needed for a certain θ.

Filtered Back-Projection

- Before IFT, $|k_\xi|$ is multiplied.
- Since this factor $|k_\xi|$ is considered as a filter in the spectral domain, the method based on FT is called
  Filtered Back-projection (FBP).
- In the actual computation, $k_\xi \in [-\infty, \infty] \to [-k_{\max}, +k_{\max}]$.
- $k_{\max}$ is Nyquist frequency determined by sampling interval.

- To avoid ringing artifact caused by high frequency component, another filter can be applied.
  (e.g. Shepp-Logan filter)

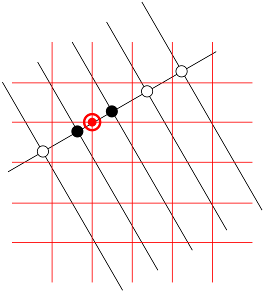
$$H(k_\xi) = \frac{2k_{\max}}{\pi} \sin\left|\frac{\pi k_\xi}{2k_{\max}}\right|$$



Handling of discrete data

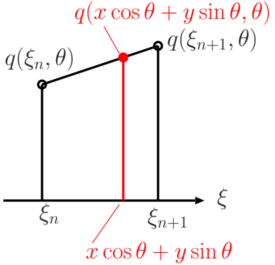
- $p(\xi_n, \theta_m) = p(n\Delta\xi, m\Delta\theta)$

  (n, m) : Integer
- $\kappa(x_i, y_j) = \kappa(i\Delta x, j\Delta y)$ (i, j) : Integer



- In order to evaluate $q(x\cos\theta + y\sin\theta, \theta)$ from $q(\xi_n, \theta)$ interpolation are needed.

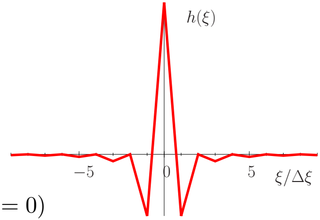
e.g. (Interpolation for ξ)



Reconstruction using convolution

$$\left\{ \begin{array}{l} c(x) = \int a(x')b(x-x')\,dx' \\ C(k) = A(k)B(k) \end{array} \right\} \Leftrightarrow \quad c(x) = \frac{1}{2\pi}\int A(k)B(k)e^{+jkx}\,dk$$

$$q(\xi,\theta) = \frac{1}{2\pi}\int_{-k_{\max}}^{k_{\max}} P(k_\xi,\theta)H(k_\xi)e^{+jk_\xi\xi}\,dk_\xi = \int_{-\xi_{\max}}^{\xi_{\max}} p(\xi',\theta)h(\xi-\xi')\,d\xi'$$



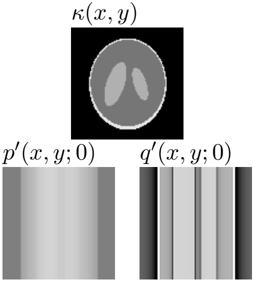
- If $H(k_\xi) = |k_\xi|$,

$$h(\xi) = \frac{1}{2\pi}\int_{-k_{\max}}^{k_{\max}} |k_\xi|e^{+jk_\xi\xi}\,dk_\xi$$

$$= \begin{cases} \dfrac{1}{2\pi}k_{\max}^2 & (\xi = 0) \\ \dfrac{1}{\pi}\dfrac{k_{\max}}{\xi}\sin(k_{\max}\xi) + \dfrac{1}{\xi^2}\left(\cos(k_{\max}\xi) - 1\right) & (\xi \neq 0) \end{cases}$$

Subtract by neighbors ⇔ Edge Enhancement

1. Sinogram : $p(\xi, \theta)$
2. Convolution :
$$q(\xi, \theta) = \int p(\xi', \theta) h(\xi - \xi') \, d\xi'$$
3. Back-projection :
$$\kappa(x, y) = \frac{1}{2\pi} \int_0^{\pi} q(x \cos\theta + y \sin\theta, \theta) \, d\theta$$



$\left(\begin{array}{l} p', q' \text{ : contrast is enhanced to display} \\ (\mathrm{erf}(I/\sigma)) \end{array} \right)$

Only one convolution for each θ.
No FT.

Number of multiplications for each θ		
Fourier transform	DFT × 2	$2N^2$
	FFT × 2	$2N \log N$
Convolutinal integral	All points	N^2
	Neighboring M pts. $(M \ll N)$	MN

$\rightarrow$ Faster computation than DFT, if the convolution is applied to neighboring points only.

Filtered Back-projection and Simple BP

- Filtered Back-Projection

$$\kappa(x,y) = \frac{1}{2}\left\langle \mathcal{F}^{-1}_{k_\xi}\left\{\mathcal{F}_\xi\left\{p(\xi,\theta)\right\}_\xi H(k_\xi)\right\}_{k_\xi}\right\rangle_\theta$$

- Simple Back-projection

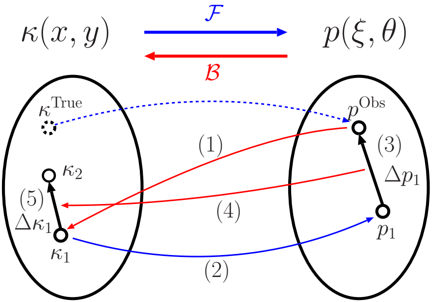
$$\begin{aligned} H(k) &= 1 \\ \kappa(x,y) &= \frac{1}{2}\left\langle \mathcal{F}^{-1}_{k_\xi}\left\{\mathcal{F}_\xi\left\{p(\xi,\theta)\right\}_\xi\right\}_{k_\xi}\right\rangle_\theta = \frac{1}{2}\left\langle p(\xi,\theta)\right\rangle_\theta \\ &= \frac{1}{2\pi}\int_0^\pi p(\xi,\theta)\,d\theta \end{aligned}$$

  - No need to FT$\rightarrow$ Fast
  - The reconstructed distribution is blurred.

$\rightarrow$ Iterate two procedures of projection and back-projection.

9.7 Iterative Reconstruction

Applying the forward projection (FP) to the reconstructed field obtained by the backward projection (BP), we can evaluate the error.
The BP of the error is added to the field obtained in the previous step. The simple BP is used for the BP algorithm, since the simple BP is fast.



(1) BP for the proj. : $\kappa_1 = \mathcal{B}\left\{p^{\mathrm{Obs}}\right\}$

(2) FP for the field : $p_1 = \mathcal{F}\left\{\kappa_1\right\}$

(3) Under-estimation : $\Delta p_1 = p^{\mathrm{Obs}} - p_1$

(4) BP for the under-est. : $\Delta\kappa_1 = \mathcal{B}\left\{\Delta p_1\right\}$

(5) Update the field : $\kappa_2 = \kappa_1 + \alpha\Delta\kappa_1$

(α : Relaxation factor for stable reconstruction $0 < \alpha \leq 1$)

Simple Back-projection

- Sinogram

$$p(\xi_n, \theta_m) = \int_{L_{nm}} \kappa(x, y)\, dl \simeq \overline{\kappa(x, y \in L_{nm})} \Delta l_{nm}$$

$$\rightarrow \overline{\kappa(x, y \in L_{nm})} = \frac{p(\xi_n, \theta_m)}{\Delta l_{nm}} \qquad \left(\Delta l_{nm} = \int_{L_{nm}} dl \right)$$

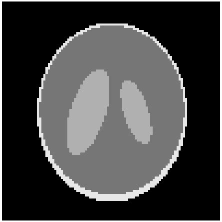
- Fraction of projection is mapped onto the internal distribution.

$$\kappa(x_i, y_j) = \frac{1}{N_n} \sum_m \sum_n a_{i,j,n,m} \frac{p(\xi_n, \theta_m)}{\Delta l_{nm}}$$

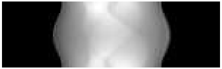
$a_{i,j,n,m}$: Overlap area fraction between the pixel (x_i, y_j) and the beam L_{nm} with width $\Delta\xi$.

9.8 Example of reconstruction by simulation

True $\kappa(x, y)$



Sinogram $p(\xi, \theta)$

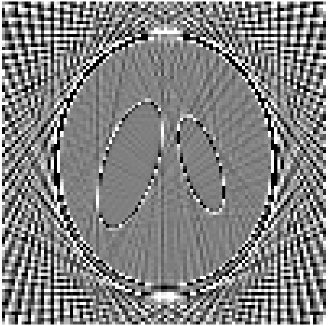


$\kappa(x, y) \in [0, 2]$,
$N_x = N_y = 100$,
$N_\xi = 100$,
$N_\theta = 45 (\Delta\theta = 4\text{deg})$

Filtered Back-Projection (Filter : Ramp function)

$\kappa(x, y)$ [0,2]

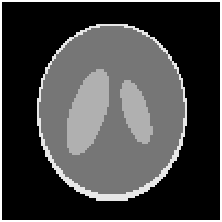
$\Delta\kappa(x, y)$ [-0.2,+0.2]



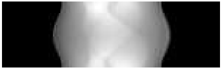
$$||\Delta\kappa||_2 \equiv \sqrt{\langle \Delta\kappa^2 \rangle} = 0.15$$

Line artifact

True $\kappa(x, y)$



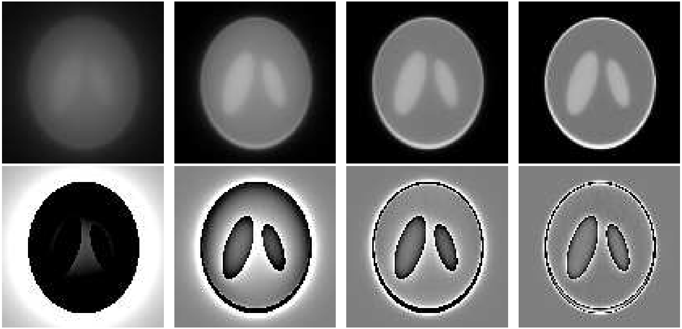
Sinogram $p(\xi, \theta)$



$\kappa(x, y) \in [0, 2]$,
$N_x = N_y = 100$,
$N_\xi = 100$,
$N_\theta = 45 (\Delta\theta = 4\text{deg})$

Iterative reconstruction

$N = 1$ | $N = 2$ | $N = 10$ | $N = 20$



$||\Delta\kappa||_2 = 0.42$ | $||\Delta\kappa||_2 = 0.25$ | $||\Delta\kappa||_2 = 0.17$ | $||\Delta\kappa||_2 = 0.11$

Reduction of edge blurring