# 8.2 Wiener deconvolution

- Model of Observation :

$$y(t) = h(t) * \widehat{x}(t) + n(t) \quad (1)$$

$(y(t), h(t)$ : known$)$

$$Y(f) = H(f) \cdot \widehat{X}(f) + N(f) \quad (2)$$

$$\left(\widehat{X} = \frac{Y-N}{H}\right)$$

$$|N(f)| \sim |N'(f)| \quad (|N'(f)| : \text{known}) \quad (3)$$

- Estimation $\widetilde{x}$ :

$$\widetilde{X}(f) = \Psi_x(f) Y(f) \qquad (\Psi_x \in \mathbb{C}) \quad (4)$$

$$\text{minimize } E = \int \underbrace{\left|\widetilde{X} - \widehat{X}\right|^2}_{=I} df \quad (5)$$

$$\rightarrow \frac{\partial E}{\partial \Psi_x} = 0 \quad (6)$$

$$\begin{aligned} I &= \left|\widetilde{X} - \widehat{X}\right|^2 = \left|\Psi_x Y - \frac{Y-N}{H}\right|^2 \\ &= \left|\left(\Psi_x - \frac{1}{H}\right) Y + \frac{N}{H}\right|^2 \\ &= \left|\left(\Psi_x - \frac{1}{H}\right) Y\right|^2 + \frac{|N|^2}{|H|^2} \\ &\quad + \left(\Psi_x - \frac{1}{H}\right) Y \frac{N^*}{H^*} + \left(\Psi_x^* - \frac{1}{H^*}\right) Y^* \frac{N}{H} \end{aligned}$$

$$\left( \begin{aligned} &\left(\Psi_x - \tfrac{1}{H}\right) Y \tfrac{N^*}{H^*} \qquad \text{(Eq.(2))} \\ &= \left(\Psi_x - \tfrac{1}{H}\right) (H\widehat{X} + N) \tfrac{N^*}{H^*} \\ &= \underbrace{\left(\Psi_x - \tfrac{1}{H}\right) \tfrac{H}{H^*} \widehat{X} N^*}_{\text{Integral}=0} + \left(\Psi_x - \tfrac{1}{H}\right) \tfrac{1}{H^*} |N|^2 \end{aligned} \right)$$

$$E = \int I(f)\, df = \int I'(f)\, df$$

$$I' = \left|\left(\Psi_x - \frac{1}{H}\right) Y\right|^2 - \frac{|N|^2}{|H|^2} + \Psi_x \frac{|N|^2}{H^*} + \Psi_x^* \frac{|N|^2}{H}$$

# Wiener deconvolution

$$\frac{\partial E}{\partial \Psi_x} = 0 \text{ or } \frac{\partial E}{\partial \Psi_x^*} = 0$$

$$E = \int I'(f)\,df$$

$$I' = \left|\left(\Psi_x - \tfrac{1}{H}\right) Y\right|^2 - \tfrac{|N|^2}{|H|^2} + \Psi_x \tfrac{|N|^2}{H^*} + \Psi_x^* \tfrac{|N|^2}{H}$$

$$\left(\begin{array}{l} \frac{\partial}{\partial \Psi_x}\left(\left|\left(\Psi_x - \frac{1}{H}\right) Y\right|^2\right) \\ \quad = \frac{\partial}{\partial \Psi_x}\left(\left(\Psi_x^* - \frac{1}{H^*}\right)\left(\Psi_x - \frac{1}{H}\right)|Y|^2\right) \\ \quad = \left(\Psi_x^* - \frac{1}{H^*}\right)|Y|^2 \end{array}\right)$$

$$\frac{\partial I'}{\partial \Psi_x^*} = \left(\Psi_x - \tfrac{1}{H}\right)|Y|^2 + \tfrac{|N|^2}{H} = 0$$

$$\rightarrow \quad \Psi_x = \frac{1}{H} \underbrace{\frac{|Y|^2 - |N|^2}{|Y|^2}}_{=\Phi_y \text{ if } |N|=|N'|.}$$

Wiener deconvolution :

$$\Psi_x(f) = \frac{\Phi_y(f)}{H(f)} \tag{7}$$

$$\Phi_y(f) = \frac{|Y|^2 - |N'(f)|^2}{|Y(f)|^2} \tag{8}$$

$$\widetilde{X}(f) = \Psi_x(f)Y(f) = \frac{1}{H(f)}\Phi_y(f)Y(f) \tag{9}$$

Spectrum of Wiener deconvolution is equivalent to the divided spectrum of the spectrum applied Wiener filter to $Y(f)$.

# In the ideal case of $N'(f) = N(f)$

$$
\begin{aligned}
&(N' = N)\\
&\Psi_x = \tfrac{1}{H}\Phi_y\\
&\quad = \tfrac{1}{H}\tfrac{|Y|^2-|N|^2}{|Y|^2} = \tfrac{1}{H}\tfrac{\left|\widehat{Y}\right|^2}{\left|\widehat{Y}\right|^2+|N|^2}\\
&\quad = \tfrac{H^*\left|\widehat{X}\right|^2}{|H|^2\left|\widehat{X}\right|^2+|N|^2}
\end{aligned} \qquad (10)
$$

- $H \neq 0$ and $N = 0$:
  $\Psi_x = \frac{1}{H} \quad \rightarrow \quad \widetilde{X} = \frac{Y}{H} = \widehat{X}$
  (Ideal case)

- $H = 0$ and $N \neq 0$:
  $\Psi_x = 0 \quad \rightarrow \quad \widetilde{X} = 0$
  (Cannot restore)

- $H = 0$ and $N = 0$:

$$
\lim_{H\to 0}\left\{\lim_{N\to 0}\left|\frac{H^*\left|\widehat{X}\right|^2}{|H|^2\left|\widehat{X}\right|^2+|N|^2}\right|\right\} = \lim_{H\to 0}\left\{\left|\frac{1}{H}\right|\right\} = \infty
$$

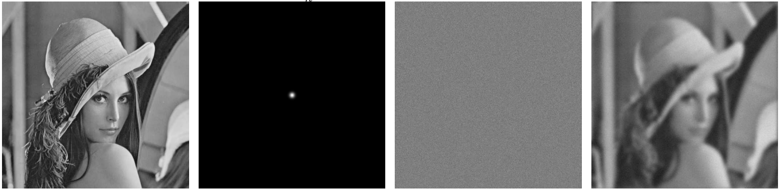
$$
\lim_{N\to 0}\left\{\lim_{H\to 0}\left|\frac{H^*\left|\widehat{X}\right|^2}{|H|^2\left|\widehat{X}\right|^2+|N|^2}\right|\right\} = 0
$$

Estimation is impossible.
(It takes different values, if the path to take limitation is different.)

> If $H = 0$, $\widetilde{X}$ cannot be detemined.
> $\rightarrow$ (Consider as $\widetilde{X}(f) = 0$ to obtain $\widetilde{x}(t)$.)

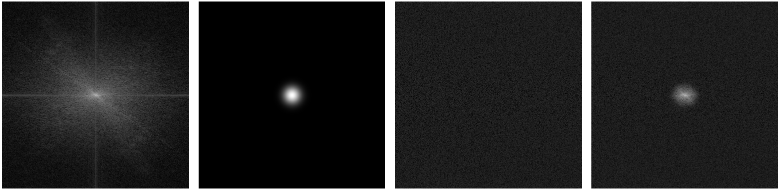
# Example of Wiener deconvolution (Input : $\sigma_h = 5$, $\sigma_n = 5$)

$\widehat{x}(\boldsymbol{r})$ | $h(\boldsymbol{r}) = \frac{1}{2\pi\sigma_h^2} e^{-\frac{r^2}{2\sigma_h^2}}$ | $n(\boldsymbol{r})$ | $y(\boldsymbol{r}) = h * \widehat{x} + n$



$[0, 255]$ | $[0, 1/(2\pi\sigma_h^2)]$ | $[-64, 64]$ | $[0, 255]$

$\widehat{X}(\boldsymbol{k})$ | $H(\boldsymbol{k}) = e^{-\frac{\sigma_h^2 k^2}{2}}$ | $|N(\boldsymbol{k})|$ | $Y(\boldsymbol{k}) = H\,\widehat{X} + N$

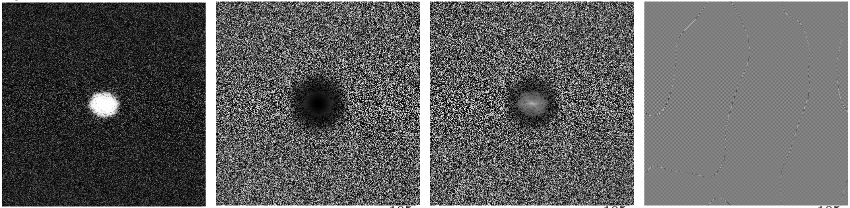


log $[1, 10^4]$ | $[0, 1]$ | log $[1, 10^4]$ | log $[1, 10^4]$

# Example of Wiener deconvolution (Result (failed case))

$(|N'(f)| = \sigma'_n = 5)$

$\Phi_y(\boldsymbol{k})$ $\Psi_x(\boldsymbol{k})$ $\widetilde{X}(\boldsymbol{k})$ $\widetilde{x}(\boldsymbol{r})$



$[0,1]$ $\log\ [1, 10^{105}]$ $\log\ [1, 10^{105}]$ $[0, 10^{105}]$

Since $|\Psi_x(\boldsymbol{k})|$ includes very large spectrum, $\widetilde{X}(\boldsymbol{k})$ diverges, and $\widetilde{x}(\boldsymbol{r})$ diverges.

# Reason of large filter gain.

- $H = 0$ and $N' \neq N$

$$\Psi_x = \lim_{H\to 0} \frac{1}{H}\Phi_y$$

$$= \lim_{H\to 0} \frac{1}{H} \frac{\mathrm{Pos}\left\{|Y|^2 - |N'|^2\right\}}{|Y|^2}$$

$$= \lim_{H\to 0} \frac{1}{H} \frac{\mathrm{Pos}\left\{\left|H\widehat{X}+N\right|^2 - |N'|^2\right\}}{\left|H\widehat{X}+N\right|^2}$$

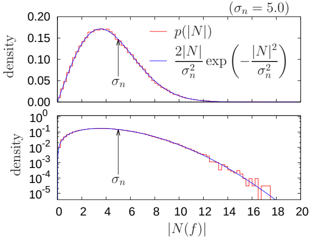
$$= \lim_{H\to 0} \frac{1}{H} \frac{\mathrm{Pos}\left\{|N|^2 - |N'|^2\right\}}{|N|^2}$$

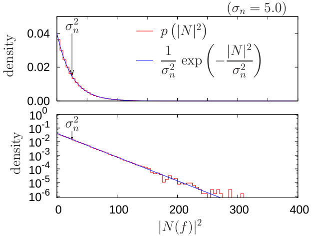
$$\mathrm{Pos}\{F\} \equiv \max\{F, 0\}$$

$$|\Psi_x| = \begin{cases} \infty & (|N| > |N'|) \\ 0 & (|N| \leq |N'|) \end{cases}$$

$\rightarrow$ Not continuous

Prob. dens. of $N(f)$



# Methods to supress divergence of filter gain.

- Specify large $|N'|$

$$\Psi'_x = \frac{1}{H}\frac{\mathrm{Pos}\left\{|Y|^2 - \alpha|N'|^2\right\}}{|Y|^2} \qquad (\alpha > 1)$$

- Limitation of domain

$$\Psi'_x(\boldsymbol{k}) = \Theta\left\{|k| \le k_{\max}\right\}\Psi_x(\boldsymbol{k})$$

$$\left(\Theta\{\mathsf{C}\} \equiv \begin{cases} 1 & (\text{C is true}) \\ 0 & (\text{C is false}) \end{cases}\right)$$

- Limitation of range

$$\Psi'_x = \Theta\left\{|\Psi(k)| \le \Psi_{\max}\right\}\Psi_x(\boldsymbol{k})$$

- Limitation of $|H|$

$$\Psi'_x = \Theta\left\{|H| > H_{\min}\right\}\Psi_x$$

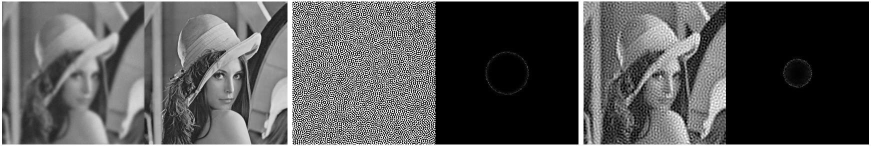
- Inspection of neighbores

$$\Psi'_x(\boldsymbol{k}) = \Theta\left\{\frac{M_0(\boldsymbol{k})}{M_a(\boldsymbol{k})} > r_{0_{\min}}\right\}\Psi_x(\boldsymbol{k})$$

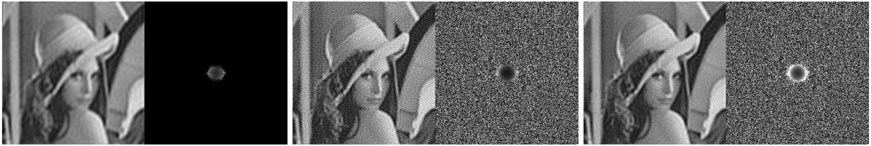
  - $M_a(\boldsymbol{k})$ is the number of pixels negiboring $\boldsymbol{k}$.
  - $M_0(\boldsymbol{k})$ is the number of pixels with $\Phi_y(\boldsymbol{k}') = 0$ within the pixels negiboring $\boldsymbol{k}$.

# Suppression of divergence in Wiener Deconvlution (e.g.)

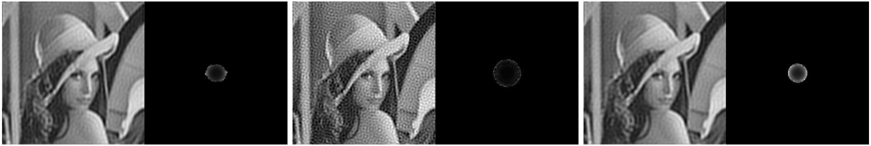
Before Deconv. | True | $k_{\max} = 0.15$ : Divergent | $k_{\max} = 0.10$ : Periodic Pattern



$\alpha = 4$ : Blurring | $\Psi_{\max} = 10$ : Noisy | $\Psi_{\max} = 5$ : Noisy



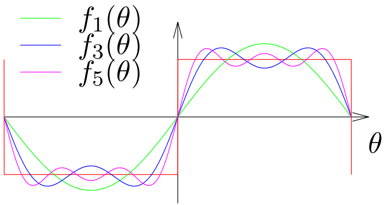
Insp. Neigh. : Ringing | $H_{\min} = 0.01$ : Peri. Patt. | $H_{\min} = 0.1$ : Peri. Patt., Ring.



# Reason of errors in Wiener deconvolution

- Divergent
  Insufficient reduction of filter gain for small $H$.
- Blurring
  Over filtering of high freq. component.
- Periodic Pattern
  Insufficient filtering for a certain component.
- Noisy
  Insufficient filtering of high freq. component.

- Ringing
  Ghost appeares along edges.
  Caused by Fourier transform wiht finite terms' trancation.
  $\Leftrightarrow$ Gibbs phenomenon
  (Impossible to avoid ringing.)



# 8.3 Estimation of Response function $H(f)$

$$Y(f) = H(f) \cdot \widehat{X}(f) + N(f)$$
$$\widetilde{X}(f) = \frac{1}{H(f)} \Phi_y(f) Y(f)$$

$H(f)$ : Function is known, but parameter of the function is unknown.

(e.g.: $H(f) = e^{-\frac{f^2}{2\sigma_f^2}}$ ($\sigma_f$ is unknown))

- Determine the parameter by least square fitting, and compute $\widetilde{H}(f)$
- The data for the least square fitting is flatten data by noise probability density function in the domain where the noise is dominant.

$$\widetilde{X}'(f) = \frac{1}{\widetilde{H}(f)} \Phi_y(f) Y(f)$$

# Least square fitting to Gaussian function

- Fitting function (Estimated Value)

$$\widetilde{f}(x_i;\, a, b) = a e^{-b x_i^2}$$

- Observed value

$$(x_i, f_i) \quad i \in \{1, \cdots, N\}$$

- Residual $\quad \Delta f_i = f_i - \widetilde{f}(x_i)$
- Average of Square Residual

$$E = \overline{(\Delta f)^2}$$

- minimize $E$

- In general method, the normal equations become non-linear equations.

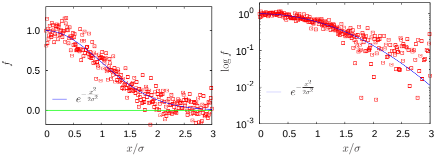
  $\rightarrow$Complex

- Mapping from $f$ to $F = \log f$.

$$\widetilde{F}_i(x_i; a, b) = \log a - b x_i^2$$

  - quadratic equation
  - Expacted error for $f$ and that for $F$ are different.
    $\rightarrow$ It should be consider the weight.

$$E = \overline{f^2 (\Delta \log f)^2}$$



# Weighted least square method

Weight $\equiv$ reliability of data

$$E = \overline{w_i \left( f_i - \widetilde{f_i} \right)^2}$$

Low reliability of the data with the large scattering.

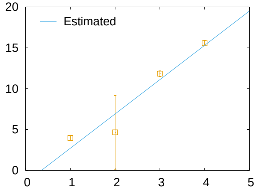
$$\mathrm{E}\left[ \left( f_i - \widetilde{f_i} \right)^2 \right] \sim \mathrm{E}\left[ \left( f_i - \widehat{f_i} \right)^2 \right] = \sigma_{f_i}^2$$

$$\rightarrow \frac{1}{\sigma_{f_i}^2} \mathrm{E}\left[ \left( f_i - \widetilde{f_i} \right)^2 \right] \sim 1(\text{constant})$$

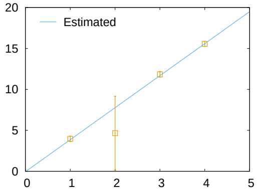
$$\rightarrow \text{Independent of } i \quad \rightarrow \quad \therefore w_i = \frac{1}{\sigma_{f_i}^2}$$

$$E = \overline{\frac{1}{\sigma_{f_i}^2} \left( f_i - \widetilde{f_i} \right)^2} \qquad (1)$$

- Without weight




- With weight



# Mapping of variable in least square method

$$E = \overline{\frac{1}{\sigma_{f_i}^2}\left(f_i - \widetilde{f_i}\right)^2} \qquad (2)$$

In the case of $F_i = F(f_i)$,

$$f_i - \widetilde{f_i} = \Delta f_i = \Delta F_i \frac{\Delta f_i}{\Delta F_i}$$

$$\simeq (F_i - \widetilde{F_i}) \left.\frac{df}{dF}\right|_i$$

$$E = \overline{\frac{1}{\left(\frac{dF}{df}\right)_i^2 \sigma_{f_i}^2}\left(F_i - \widetilde{F_i}\right)^2} \qquad (3)$$

$\left|\frac{dF}{df}\right|$ shows a magnification factor of error bar.

If $F = \log f$,

$$\frac{dF}{df} = \frac{1}{f}$$

$$E = \overline{\frac{f_i^2}{\sigma_{f_i}^2}(\log f_i - \log \widetilde{f_i})^2}$$

- $f = \widehat{f} + n$
  $n$: white $\rightarrow \sigma_n^2 = \text{const.}$
- $E = \frac{1}{\sigma_n^2}\overline{f_i^2 (\log f_i - \log \widetilde{f_i})^2}$
- Larger $f$ has larger weight.