

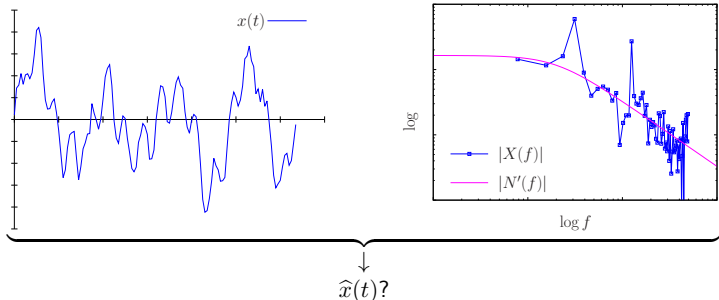
8. Noise Reduction using Spectrum

8.1 Wiener filter

Model of Observation : $x(t) = \hat{x}(t) + n(t)$ (1)

$x(t)$: Observed (known)	$X(f)$ (known)	$ X(f) $ (known)
$\hat{x}(t)$: True (Unknown)	$\hat{X}(f)$ (Unknown)	$ \hat{X}(f) $ (Unknown)
$n(t)$: Noise (Unknown)	$N(f)$ (Unknown)	$ N(f) $ (unknown), $ N'(f) $ (known)

($|N'(f)|$: substitute of $|N(f)|$,
white/red, $\sigma_{n'}^2, \dots$)



Parseval's theorem

Parseval's theorem

$$\int_{-\infty}^{\infty} |a(t)|^2 dt = \int_{-\infty}^{\infty} |A(f)|^2 df \quad (2)$$

$$\begin{cases} A(f) = \int_{-\infty}^{\infty} a(t) e^{-j2\pi ft} dt \\ a(t) = \int_{-\infty}^{\infty} A(f) e^{+j2\pi ft} df \end{cases} \quad (3)$$

(Proof)

$$\begin{aligned} \text{LHS} &= \int_t \left(\int_f A(f) e^{+j2\pi ft} df \cdot \int_{f'} A^*(f') e^{-j2\pi f't} df' \right) dt \\ &= \int_f \int_{f'} A(f) A^*(f') \underbrace{\int_t e^{+j2\pi(f-f')t} dt}_{=\delta(f-f')} df' df \\ &= \int_f \int_{f'} A(f) A^*(f') \delta(f - f') df' df \\ &= \int_f A(f) A^*(f) df = \int_{-\infty}^{\infty} |A(f)|^2 df = \text{RHS} \end{aligned}$$

Spectral product of Non-Correlated Signals

If $a(t)$ is independent $n(t)$

$$\int A^*(f)N(f) df = C_{a,n}(0) = 0. \quad (4)$$

Integral of spectral product of Non-Correlated Signals vanishes.

Proof.

$$\begin{aligned} & \int A^*(f)N(f) df \quad (\text{Express } A(f) \text{ and } N(f) \text{ by FT}) \\ &= \int_f \int_t a^*(t)e^{+j2\pi ft} dt \int_{t'} n(t')e^{-j2\pi ft'} dt' df \quad (\text{Exchange the order}) \\ &= \int_t \int_{t'} a^*(t)n(t') \int_f e^{+j2\pi f(t-t')} df dt' dt \quad (\int e^{+j2\pi f(t-t')} df = \delta(t-t')) \\ &= \int_t a^*(t) \int_{t'} n(t')\delta(t-t') dt' dt = \int_t a^*(t)n(t) dt = C_{a,n}(0) = 0. \end{aligned}$$

Filtering function in Spectral domain $\Phi_x(f)$

$$x(t) = \hat{x}(t) + n(t) \quad (x, \hat{x}, n \in \mathbb{R}) \quad (1)$$

$$X(f) = \hat{X}(f) + N(f) \quad (X, \hat{X}, N \in \mathbb{C}) \quad (5)$$

$$\tilde{X}(f) = X(f)\Phi_x(f) \quad (6)$$

$$\tilde{x}(t) = \mathcal{F}^{-1} \left\{ \tilde{X}(f) \right\} \quad (7)$$

$\Phi_x(f) \in \mathbb{R}$: Filtering function

$\tilde{X}(f) \in \mathbb{C}$: Estimated spectrum

$\tilde{x}(t)$: Estimated signal

- Determine $\tilde{x}(t)$ so that the residual is minimized.

(Integral of residual)

$$E = \int |\tilde{x}(t) - \hat{x}(t)|^2 dt \quad (8)$$

- From the Parseval's theorem

$$\begin{aligned} E &= \int |\tilde{x}(t) - \hat{x}(t)|^2 dt \\ &= \int \underbrace{|\tilde{X}(f) - \hat{X}(f)|^2}_{\equiv I(f) \geq 0} df \quad (9) \end{aligned}$$

- Since integrand $I(f) \geq 0$,
Minimize $E \Leftrightarrow$ Minimize $I(f)$ for all f .
 $\rightarrow \frac{\partial E}{\partial \Phi_x} = 0 \quad (10)$

(Stationary condition)

- $E = \int |\tilde{X} - \hat{X}|^2 df = \int |X\Phi_x - \hat{X}|^2 df$
 E is the function of the function Φ_x
This is called functional.

$$\begin{aligned}
 I &= \left| \tilde{X} - \hat{X} \right|^2 = \left| X\Phi_x - \hat{X} \right|^2 \\
 &= \left| (\hat{X} + N)\Phi_x - \hat{X} \right|^2 = \left| \hat{X}(\Phi_x - 1) + N\Phi_x \right|^2 \\
 &= (\hat{X}(\Phi_x - 1) + N\Phi_x)^* (\hat{X}(\Phi_x - 1) + N\Phi_x) \\
 &= \underbrace{\left| \hat{X} \right|^2 (\Phi_x - 1)^2 + |N|^2 \Phi_x^2}_{\equiv I'} \\
 &\quad + \underbrace{(\hat{X}^* N + \hat{X} N^*)(\Phi_x - 1)\Phi_x}_{\text{Integral over } f \text{ vanishes. } (\because \text{Eq.(4)})}
 \end{aligned}$$

$$E = \int I df = \int I' df$$

$$I' = \left| \hat{X} \right|^2 (\Phi_x - 1)^2 + |N|^2 \Phi_x^2 \geq 0$$

$$\text{minimize } E \Leftrightarrow \text{minimize } I' \Leftrightarrow \frac{\partial I'}{\partial \Phi_x} = 0$$

$$\begin{aligned}
 \frac{\partial I'}{\partial \Phi_x} &= 2 \left(\left| \hat{X} \right|^2 (\Phi_x - 1) + |N|^2 \Phi_x \right) = 0 \\
 \rightarrow \Phi_x &= \frac{|\hat{X}|^2}{|\hat{X}|^2 + |N|^2}
 \end{aligned}$$

Wiener filter

$$\Phi_x(f) = \frac{|\hat{X}(f)|^2}{|\hat{X}(f)|^2 + |N(f)|^2} \quad (11)$$

$$(0 \leq \Phi_x(f) \leq 1)$$

However, this form includes FT of true solution, $\hat{X}(f)$.

Representation of filter function using observed value

$$\begin{aligned}
 & \begin{pmatrix} X = \hat{X} + N \\ \tilde{X} = X\Phi_x \end{pmatrix} \\
 I &= \left| \tilde{X} - \hat{X} \right|^2 = |X\Phi_x - (X - N)|^2 \\
 &= |X(\Phi_x - 1) + N|^2 \\
 &= (X(\Phi_x - 1) + N)^*(X(\Phi_x - 1) + N) \\
 &= |X|^2 (\Phi_x - 1)^2 + |N|^2 \\
 &\quad + (X^*N + XN^*)(\Phi_x - 1) \\
 & \begin{pmatrix} X^*N + XN^* \\ = (\hat{X} + N)^*N + (\hat{X} + N)N^* \\ = \hat{X}^*N + \hat{X}N^* + 2|N|^2 \end{pmatrix} \\
 &= |X|^2 (\Phi_x - 1)^2 - |N|^2 + 2|N|^2 \Phi_x \\
 &\quad + (\hat{X}^*N + \hat{X}N^*)(\Phi_x - 1)
 \end{aligned}$$

Since $\int A^*N df = 0$ (Eq.(4))
the last term is removed from I .

$$\begin{aligned}
 E &= \int I df = \int I' df \\
 I' &= |X|^2 (\Phi_x - 1)^2 - |N|^2 + 2|N|^2 \Phi_x \\
 \frac{\partial I'}{\partial \Phi_x} &= 2 \left(|X|^2 (\Phi_x - 1) + |N|^2 \right) = 0 \\
 \rightarrow \Phi_x &= \frac{|X|^2 - |N|^2}{|X|^2} \simeq \frac{|X|^2 - |N'|^2}{|X|^2} \\
 (\because N' &\simeq N) \\
 &\text{(All vars. in RHS are known.)}
 \end{aligned}$$

$\therefore$

$$\Phi_x(f) = \frac{|X(f)|^2 - |N'(f)|^2}{|X(f)|^2} \quad (12)$$

Steps to Apply Wiener filter

1 $|X(f)|^2$

① Measurement of $x(t)$

② $|X(f)|^2 = |\mathcal{F}\{x(t)\}|^2$

2 $|N'(f)|^2$

► Measurement of $n(t)$
(without signal)

$$|N'(f)|^2 = |\mathcal{F}\{n'(t)\}|^2$$

► If impossible,
determine considering property of noise.

- White : $|N'(f)| = \text{const}$
- Brownian : $|N'(f)| \propto \frac{1}{f^2 + a^2}$

3 $\Phi_x(f) = \frac{|X(f)|^2 - |N'(f)|^2}{|X(f)|^2}$

► If $\Phi_x(f) < 0$, $\Phi_x(f) = 0$.
(irreversible)

Wiener filter only corrects the amplitude, it does not correct phase.

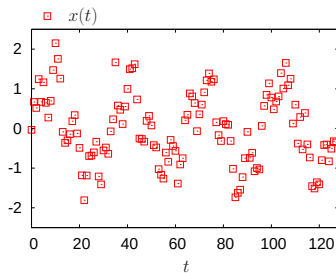
$$(\Phi_x < 0 \Leftrightarrow |\Phi_x|e^{-i\pi})$$

4 $\tilde{x}(t)$

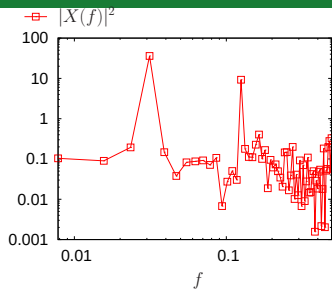
① $\tilde{X}(f) = X(f)\Phi_x(f)$

② $\tilde{x}(t) = \mathcal{F}^{-1}\{\tilde{X}(f)\}$

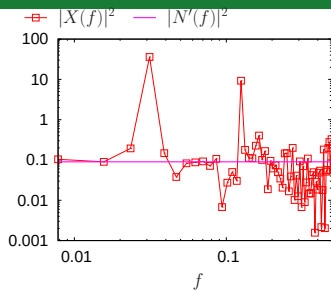
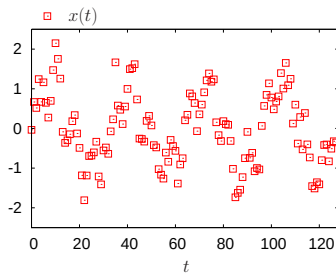
Example applying Wiener filter.



1 $x(t), X(f)$



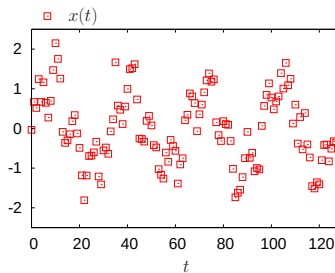
Example applying Wiener filter.



1 $x(t), X(f)$

2 $N'(f)$

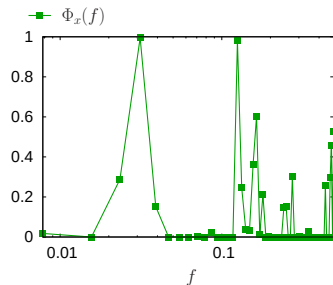
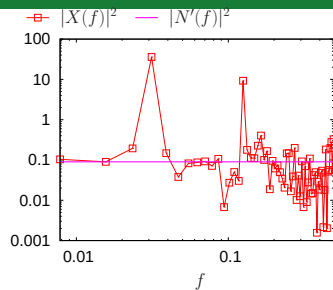
Example applying Wiener filter.



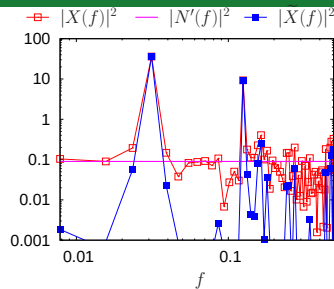
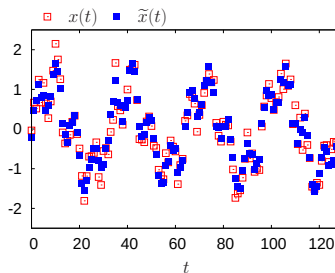
1 $x(t), X(f)$

2 $N'(f)$

3 $\Phi_x(f) = \frac{|X(f)|^2 - |N'(f)|^2}{|X(f)|^2}$



Example applying Wiener filter.

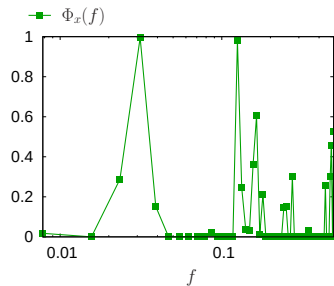


1 $x(t), X(f)$

2 $N'(f)$

3 $\Phi_x(f) = \frac{|X(f)|^2 - |N'(f)|^2}{|X(f)|^2}$

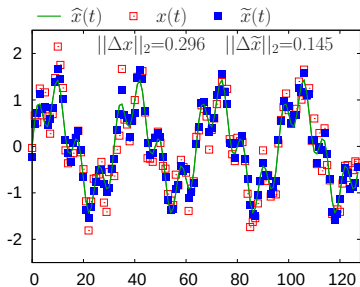
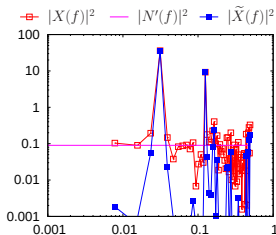
4 $\tilde{X}(f), \tilde{x}(t)$



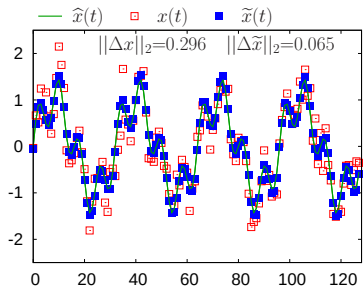
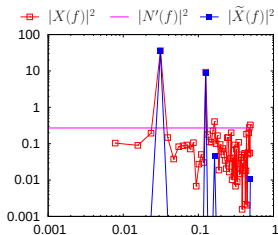
Effect of $|N'(f)|$ (White Noise)

$$x(t) = \sin\left(\frac{2\pi t}{T/4}\right) + \frac{1}{2} \sin\left(\frac{2\pi t}{T/16}\right) + n, \quad \sigma_n^2 = 0.09$$

$$|N'| = \sigma_n^2 = 0.09$$

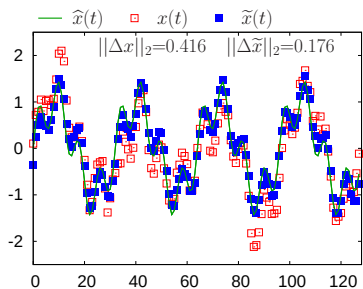
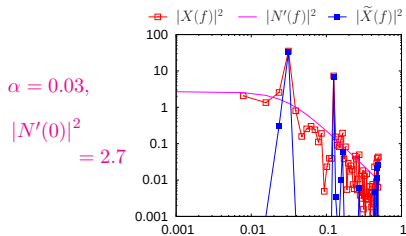
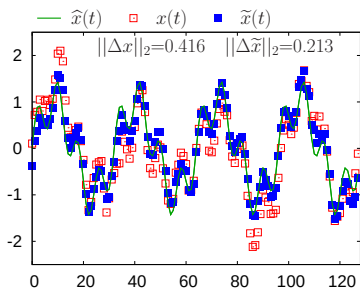
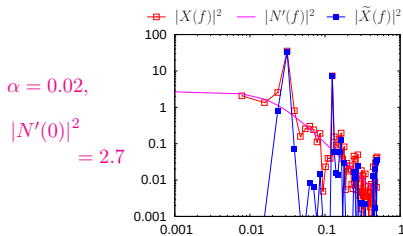


$$|N'| = 3\sigma_n^2 = 0.27$$



Effect of $|N'(f)|$ (Brownian Noise)

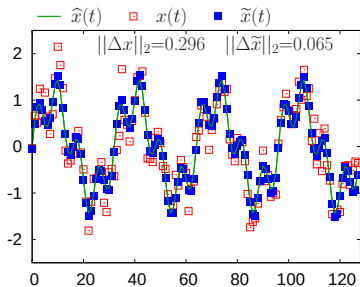
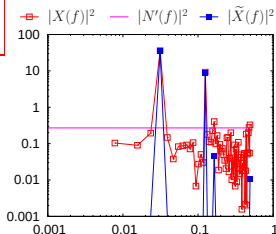
$$\left(x(t) = \sin\left(\frac{2\pi t}{T/4}\right) + \frac{1}{2} \sin\left(\frac{2\pi t}{T/16}\right) + r, \quad \alpha = 0.02, \quad \sigma_n^2 = 0.04 \quad (|N'(0)|^2 = 2.7) \right)$$



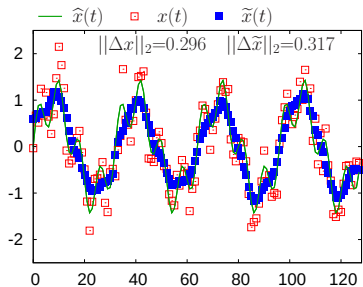
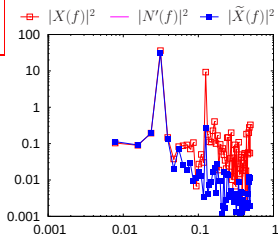
Comparison between Wiener filter and Moving average

$$x(t) = \sin\left(\frac{2\pi t}{T/4}\right) + \frac{1}{2} \sin\left(\frac{2\pi t}{T/16}\right) + n, \quad \sigma_n^2 = 0.09$$

Wiener filter
($|N'|^2 = 0.27$)



Mov. Ave
($N_m = 7$)



Summary of Wiener filter

In the case where $x(t)$ and $|N'(f)|$ is known:

$$X(f) = \mathcal{F}\{x(t)\}$$

$$\Phi_x(f) = \frac{|X(f)|^2 - |N'(f)|^2}{|X(f)|^2}$$

$$\tilde{X}(f) = X(f)\Phi_x(f)$$

$$\tilde{x}(t) = \mathcal{F}^{-1}\{\tilde{X}(f)\}$$

- Wiener filter can be taken into account of Noise spectrum.
- Wiener filter is applicable when the spectrum of signal has several peaks. This is different from the moving average.

Wiener filter is called 'Optimal filter'.