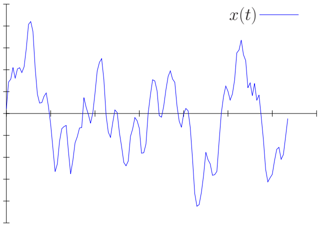
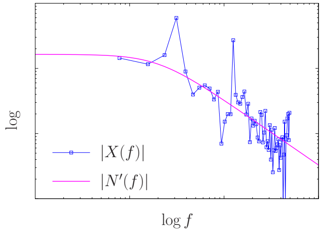
# 8. Noise Reduction using Spectrum
# 8.1 Wiener filter

Model of Observation : $$x(t) = \widehat{x}(t) + n(t) \qquad (1)$$

| | | | |
|---|---|---|---|
| $x(t)$ : Observed (known) | $X(f)$ (known) | $\lvert X(f)\rvert$ (known) |
| $\widehat{x}(t)$ : True (Unknown) | $\widehat{X}(f)$ (Unknown) | $\lvert\widehat{X}(f)\rvert$ (Unknown) |
| $n(t)$ : Noise (Unknown) | $N(f)$ (Unknown) | $\lvert N(f)\rvert$ (unknown), $\lvert N'(f)\rvert$ (known) |

$\left( \lvert N'(f)\rvert$ : sustitute of $\lvert N(f)\rvert$, white/red, $\sigma^2_{n'}$,... $\right)$




$\downarrow$

$\widehat{x}(t)$?

# Parseval's theorem

**Parseval's theorem**

$$\int_{-\infty}^{\infty} |a(t)|^2 \, dt = \int_{-\infty}^{\infty} |A(f)|^2 \, df \quad (2)$$

$$\begin{cases} A(f) = \int_{-\infty}^{\infty} a(t) e^{-j2\pi f t} \, dt \\ a(t) \ = \int_{-\infty}^{\infty} A(f) e^{+j2\pi f t} \, df \end{cases} \quad (3)$$

(Proof)

$$\begin{aligned} \text{LHS} &= \int_t \left( \int_f A(f) e^{+j2\pi f t} \, df \cdot \int_{f'} A^*(f') e^{-j2\pi f' t} \, df' \right) dt \\ &= \int_f \int_{f'} A(f) A^*(f') \underbrace{\int_t e^{+j2\pi (f-f')t} \, dt}_{=\delta(f-f')} \, df' \, df \\ &= \int_f \int_{f'} A(f) A^*(f') \, \delta(f - f') \, df' \, df \\ &= \int_f A(f) A^*(f) \, df = \int_{-\infty}^{\infty} |A(f)|^2 \, df = \text{RHS} \end{aligned}$$

# Spectral product of Non-Correlated Signals

If $a(t)$ is independent $n(t)$

$$\int A^*(f)N(f)\,df = C_{a,n}(0) = 0. \tag{4}$$

Integral of spectral product of Non-Correlated Signals vanishes.

Proof.

$$\int A^*(f)N(f)\,df \qquad \text{(Express } A(f) \text{ and } N(f) \text{ by FT)}$$

$$= \int_f \int_t a^*(t)e^{+j2\pi ft}\,dt \int_{t'} n(t')e^{-j2\pi ft'}\,dt'\,df \quad \text{(Exchange the order)}$$

$$= \int_t \int_{t'} a^*(t)n(t') \int_f e^{+j2\pi f(t-t')}\,df\,dt'\,dt \quad \left(\int e^{+j2\pi f(t-t')}\,df = \delta(t-t')\right)$$

$$= \int_t a^*(t) \int_{t'} n(t')\delta(t-t')\,dt'\,dt = \int_t a^*(t)n(t)\,dt = C_{a,n}(0) = 0.$$

# Filtering function in Sprctral domain $\Phi_x(f)$

$$x(t) = \widehat{x}(t) + n(t) \quad (x, \widehat{x}, n \in \mathbb{R}) \quad (1)$$

$$X(f) = \widehat{X}(f) + N(f) \quad (X, \widehat{X}, N \in \mathbb{C}) \quad (5)$$

$$\widetilde{X}(f) = X(f)\Phi_x(f) \quad (6)$$

$$\widetilde{x}(t) = \mathcal{F}^{-1}\left\{\widetilde{X}(f)\right\} \quad (7)$$

$\Phi_x(f) \in \mathbb{R}$ : Filtering function
$\widetilde{X}(f) \in \mathbb{C}$ : Estimated spectrum
$\widetilde{x}(t)$ : Estimated signal

- Determine $\widetilde{x}(t)$ so that the residual is minimized.

  (Integral of residual)

$$E = \int |\widetilde{x}(t) - \widehat{x}(t)|^2 \, dt \quad (8)$$

- From the Parseval's theorem

$$E = \int |\widetilde{x}(t) - \widehat{x}(t)|^2 \, dt = \int \underbrace{\left|\widetilde{X}(f) - \widehat{X}(f)\right|^2}_{\equiv I(f) \geq 0} df \quad (9)$$

- Since integrand $I(f) \geq 0$,
  Minimize $E \Leftrightarrow$ Minimize $I(f)$ for all $f$.

$$\rightarrow \quad \frac{\partial E}{\partial \Phi_x} = 0 \quad (10)$$

  (Stationary condition)

- $$E = \int \left|\widetilde{X} - \widehat{X}\right|^2 df = \int \left|X\Phi_x - \widehat{X}\right|^2 df$$

  $E$ is the function of the function $\Phi_x$
  This is called functional.

$$I = \left|\widetilde{X} - \widehat{X}\right|^2 = \left|X\Phi_x - \widehat{X}\right|^2$$
$$= \left|(\widehat{X} + N)\Phi_x - \widehat{X}\right|^2 = \left|\widehat{X}(\Phi_x - 1) + N\Phi_x\right|^2$$
$$= (\widehat{X}(\Phi_x - 1) + N\Phi_x)^*(\widehat{X}(\Phi_x - 1) + N\Phi_x)$$
$$= \underbrace{\left|\widehat{X}\right|^2 (\Phi_x - 1)^2 + |N|^2 \Phi_x^2}_{\equiv I'} + \underbrace{(\widehat{X}^* N + \widehat{X} N^*)(\Phi_x - 1)\Phi_x}_{\text{Integral over } f \text{ vanishes.} (\because \text{ Eq.}(4))}$$

$$E = \int I\, df = \int I'\, df$$

$$I' = \left|\widehat{X}\right|^2 (\Phi_x - 1)^2 + |N|^2 \Phi_x^2 \geq 0$$

minimize $E$ $\Leftrightarrow$ minimize $I'$ $\Leftrightarrow$ $\dfrac{\partial I'}{\partial \Phi_x} = 0$

$$\frac{\partial I'}{\partial \Phi_x} = 2\left(\left|\widehat{X}\right|^2 (\Phi_x - 1) + |N|^2 \Phi_x\right) = 0$$
$$\rightarrow \Phi_x = \frac{\left|\widehat{X}\right|^2}{\left|\widehat{X}\right|^2 + |N|^2}$$

**Wiener filter**

$$\Phi_x(f) = \frac{\left|\widehat{X}(f)\right|^2}{\left|\widehat{X}(f)\right|^2 + |N(f)|^2} \qquad (11)$$

$$(0 \leq \Phi_x(f) \leq 1)$$

However, this form includes FT of true solution, $\widehat{X}(f)$.

# Representation of filter function using observed value

$$\left( \begin{array}{l} X = \widehat{X} + N \\ \widetilde{X} = X\Phi_x \end{array} \right)$$

$$I = \left|\widetilde{X} - \widehat{X}\right|^2 = |X\Phi_x - (X - N)|^2$$
$$= |X(\Phi_x - 1) + N|^2$$
$$= (X(\Phi_x - 1) + N)^*(X(\Phi_x - 1) + N)$$
$$= |X|^2 (\Phi_x - 1)^2 + |N|^2$$
$$+ (X^*N + XN^*)(\Phi_x - 1)$$

$$\left( \begin{array}{l} X^*N + XN^* \\ \quad = (\widehat{X} + N)^*N + (\widehat{X} + N)N^* \\ \quad = \widehat{X}^*N + \widehat{X}N^* + 2|N|^2 \end{array} \right)$$

$$= |X|^2 (\Phi_x - 1)^2 - |N|^2 + 2|N|^2 \Phi_x$$
$$+ (\widehat{X}^*N + \widehat{X}N^*)(\Phi_x - 1)$$

Since $\int A^* N \, df = 0$ (Eq.(4))
the last term is removed from $I$.

$$E = \int I \, df = \int I' \, df$$
$$I' = |X|^2 (\Phi_x - 1)^2 - |N|^2 + 2|N|^2 \Phi_x$$
$$\frac{\partial I'}{\partial \Phi_x} = 2\left(|X|^2 (\Phi_x - 1) + |N|^2\right) = 0$$
$$\rightarrow \quad \Phi_x = \frac{|X|^2 - |N|^2}{|X|^2} \simeq \frac{|X|^2 - |N'|^2}{|X|^2}$$
$$(\because N' \simeq N)$$

(All vars. in RHS are known.)

$\therefore$

$$\Phi_x(f) = \frac{|X(f)|^2 - |N'(f)|^2}{|X(f)|^2} \tag{12}$$

# Steps to Apply Wiener filter

1. $|X(f)|^2$
   1. Measurement of $x(t)$
   2. $|X(f)|^2 = |\mathcal{F}\{x(t)\}|^2$

2. $|N'(f)|^2$
   - Measurement of $n(t)$ (without signal) $|N'(f)|^2 = |\mathcal{F}\{n'(t)\}|^2$
   - If impossible, detemine considering property of noise.
     - White : $|N'(f)| = \text{const}$
     - Brownian : $|N'(f)| \propto \frac{1}{f^2+a^2}$

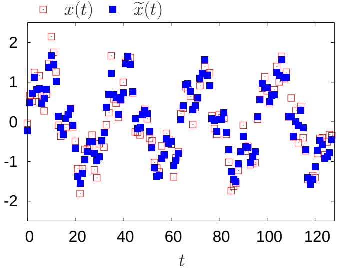
3. $\Phi_x(f) = \frac{|X(f)|^2 - |N'(f)|^2}{|X(f)|^2}$
   - If $\Phi_x(f) < 0$, $\Phi_x(f) = 0$. (irreversible)

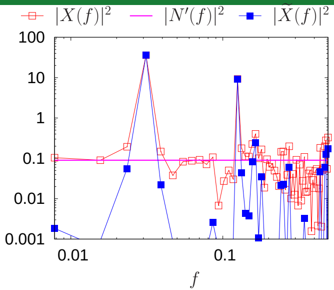
   Wiener filter only corrects the amplitude, it does not correct phase.

   $(\Phi_x < 0 \Leftrightarrow |\Phi_x| e^{-i\pi})$

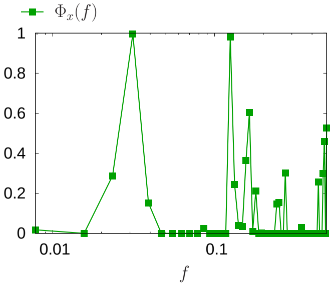
4. $\widetilde{x}(t)$
   1. $\widetilde{X}(f) = X(f)\Phi_x(f)$
   2. $\widetilde{x}(t) = \mathcal{F}^{-1}\left\{\widetilde{X}(f)\right\}$

# Example applying Wiener filter.



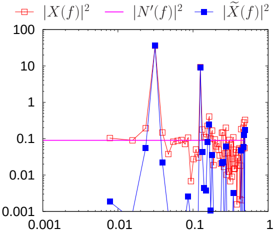

1. $x(t)$, $X(f)$
2. $N'(f)$
3. $\Phi_x(f) = \frac{|X(f)|^2 - |N'(f)|^2}{|X(f)|^2}$
4. $\widetilde{X}(f)$, $\widetilde{x}(t)$

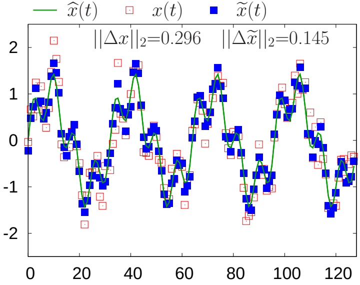


# Effect of $|N'(f)|$ (White Noise)

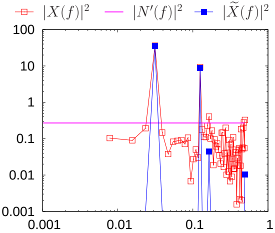
$$\left(x(t) = \sin\left(\frac{2\pi t}{T/4}\right) + \frac{1}{2}\sin\left(\frac{2\pi t}{T/16}\right) + n, \quad \sigma_n^2 = 0.09\right)$$

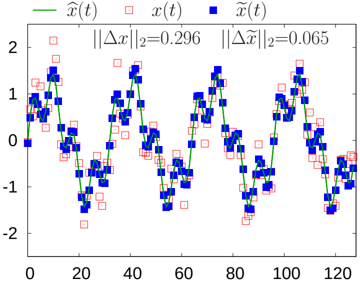
$|N'| = \sigma_n^2$
$= 0.09$




$|N'| = 3\sigma_n^2$
$= 0.27$



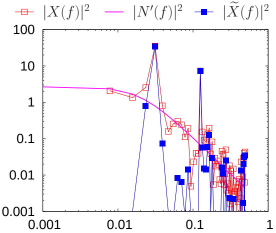



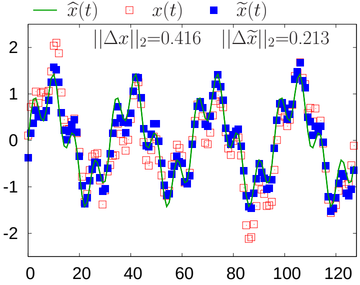
# Effect of $|N'(f)|$ (Brownian Noise)

$$\left(x(t) = \sin\left(\frac{2\pi t}{T/4}\right) + \frac{1}{2}\sin\left(\frac{2\pi t}{T/16}\right) + r, \quad \alpha = 0.02, \quad \sigma_{n'}^2 = 0.04 \quad (|N'(0)|^2 = 2.7)\right)$$

$\alpha = 0.02,$

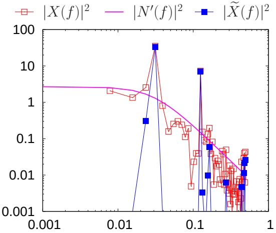
$|N'(0)|^2 = 2.7$

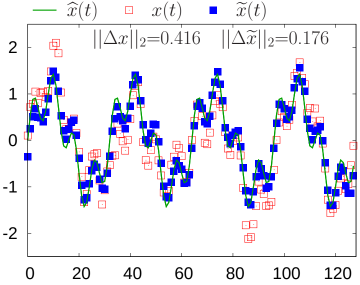



$\alpha = 0.03,$

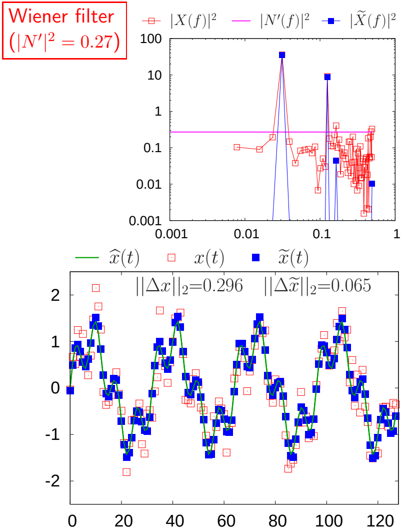
$|N'(0)|^2 = 2.7$

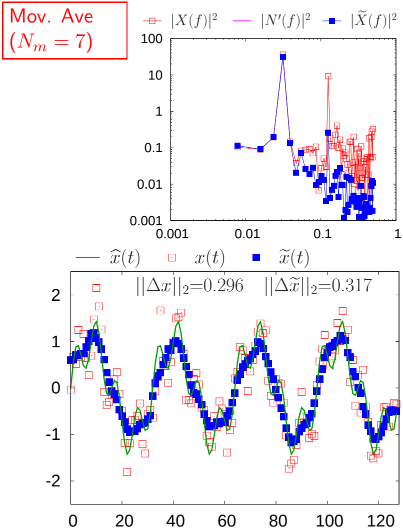


# Comparison between Wiener filter and Moving average

$$\left(x(t) = \sin\left(\frac{2\pi t}{T/4}\right) + \frac{1}{2}\sin\left(\frac{2\pi t}{T/16}\right) + n, \quad \sigma_n^2 = 0.09\right)$$



# Summary of Wiener filter

In the case where $x(t)$ and $|N'(f)|$ is known:

$$X(f) = \mathcal{F}\{x(t)\}$$
$$\Phi_x(f) = \frac{|X(f)|^2 - |N'(f)|^2}{|X(f)|^2}$$
$$\widetilde{X}(f) = X(f)\Phi_x(f)$$
$$\widetilde{x}(t) = \mathcal{F}^{-1}\left\{\widetilde{X}(f)\right\}$$

- Wiener filter can be taken into account of Noise spectrum.
- Wiener filter is applicable when the spectrum of signal has several peaks. This is different from the moving average.

Wiener filter is called ‘Optimal filter’.