Image Processing

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1. Fourier Transform (Series Expansion)

time t [s]	$\Longleftrightarrow$	frequency ω [rad/s]
position $\boldsymbol{r}$ [m]	$\Longleftrightarrow$	wave number $\boldsymbol{k}$ [rad/m]

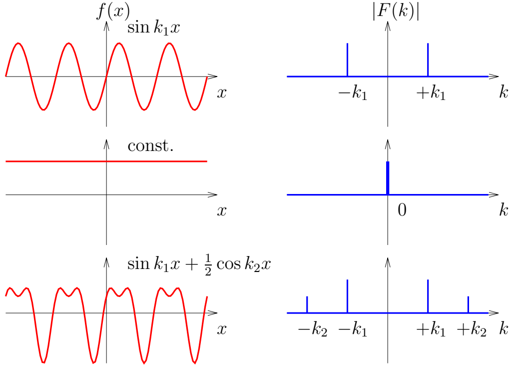
◎ Advantages of Fourier Transform

- understanding of data characteristic (period)
- understanding data propagation mechanism
  (Convolution $\leftrightarrow$ Product in Fourier space)
- Filtering (reduction of noise, enhancement of certain characteristic)
- simple mathematical operation

  ⋮

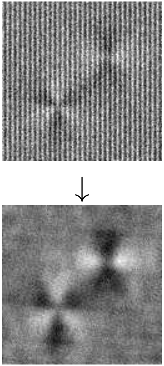
○ Data characteristic

eg.



We can understand periodicity of $f(x)$ from spectrum $|F(k)|$.

○ example of filtering



1.1 Complex Fourier Series

Complete Orthogonal System

When a complex function $f(\theta); \theta \in [-\pi, +\pi]$ satisfies

- $\int_a^b |f(\theta)|\, d\theta < \infty$
- $f(-\pi) = f(+\pi)$,

$f(\theta)$ is expressed by a series of $e^{im\theta}$.

$$f(\theta) = \sum_{m=-\infty}^{\infty} F_m e^{im\theta} \quad (1)$$

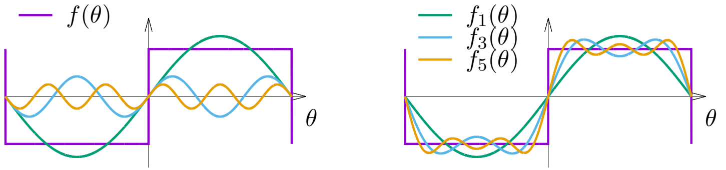
$$F_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-im\theta}\, d\theta \quad (2)$$

$(f \in \mathbb{C},\ \theta \in \mathbb{R},\ m \in \mathbb{Z}$(Integer Numbers)$)$

Completeness

$$f(\theta) = \sum_{m=-\infty}^{\infty} F_m e^{im\theta} \qquad \text{(Completeness)} \qquad (1)$$

Proof: Show $\lim_{N\to\infty} f_N(\theta) - f(\theta) = 0$. (detail is omitted)



$f_N(x)$: Approximated function by truncated finite terms

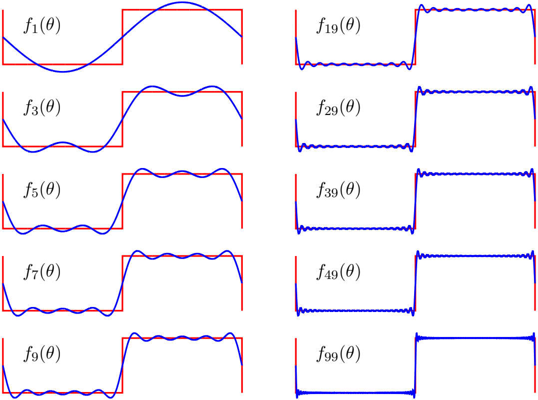
$$\left(f_N(\theta) = \sum_{m=-N}^{N} F_m e^{im\theta} \right)$$

※ Discontinuities of $f(\theta)$ between the domain, or the inconsistency at both ends ($f(-\pi) \neq f(+\pi)$) are acceptable.
In these cases, the value at the discontinuous point is considered as the average. (Dirichlet's theorem)

Gibbs phenomenon

- The truncated function with finite terms $f_N(\theta)$ has error with dumping oscillation around discontinuous point.

$$f_N(\theta) = \sum_{m=-N}^{N} F_m e^{im\theta}$$



Orthogonality

$$f(\theta) = \sum_{m=-\infty}^{\infty} F_m e^{im\theta} \qquad (1)$$

$$F_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-im\theta}\,d\theta \qquad (2)$$

We can prove Eq. (2) using the following orthogonal property.

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(m-m')\theta}\,d\theta = \delta_{m,m'} = \begin{cases} 1 & (m = m') \\ 0 & (m \neq m') \end{cases} \qquad \text{(Orthogonality)} \quad (3)$$

Scaling of domain($\theta \in [-\pi, +\pi] \rightarrow x \in [-l, l]$)

$$f(\theta) = \sum_{m=-\infty}^{\infty} F_m e^{im\theta} \quad (1)$$

$$F_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-im\theta}\, d\theta \quad (2)$$

$$\Rightarrow \left(x = \frac{l}{\pi}\theta, \qquad k_m = m\frac{\pi}{l} \in \mathbb{R} \right)$$

$$\Downarrow$$

$$f(x) = \sum_{m=-\infty}^{\infty} F_{k_m} e^{ik_m x} \quad (4)$$

$$F_{k_m} = \frac{1}{2l} \int_{-l}^{l} f(x) e^{-ik_m x}\, dx \quad (5)$$

※ Both dimensions of $f(x)$ and F_{k_m} are same.

(* Arguments of elementary functions has no dimensions. (Exception: x of $\log(x)$)
* The value of elementary function has also no dimensions.)

Shifting origin($x \in [-l, +l] \ \rightarrow x' \in [-l+a, l+a]$)

$$f(x) = \sum_{m=-\infty}^{\infty} F_{k_m} e^{ik_m x} \qquad (4)$$

$$F_{k_m} = \frac{1}{2l} \int_{-l}^{l} f(x) e^{-ik_m x}\, dx \qquad (5)$$

$\xrightarrow{x \to x+a}$

$$f(x+a) = \sum_{m=-\infty}^{\infty} F_{k_m} e^{ik_m (x+a)}$$

$$F_{k_m} = \frac{1}{2l} \int_{-l}^{l} f(x+a) e^{-ik_m (x+a)}\, dx$$

$\Downarrow \quad (x' = x + a)$

$$f(x') = \sum_{m=-\infty}^{\infty} F_{k_m} e^{ik_m x'} \qquad (6)$$

$$F_{k_m} = \frac{1}{2l} \int_{-l+a}^{l+a} f(x') e^{-ik_m x'}\, dx' \qquad (7)$$

Only the change of integration range

Fourier Transform($l \to \infty$)

$$f(x) = \sum_{m=-\infty}^{\infty} F_{k_m} e^{ik_m x}, \quad F_{k_m} = \frac{1}{2l} \int_{-l}^{l} f(x) e^{-ik_m x} \, dx \qquad \left(k_m = m\frac{\pi}{l} = m\Delta k\right)$$

If $l \to \infty$, then $\Delta k \to 0$, discrete $k_m \to$ continuous k, , $\sum_{m=-\infty}^{\infty} \cdots \Delta k \to \int_{-\infty}^{\infty} \cdots \, dk$

$$f(x) = \lim_{\substack{l\to\infty, \\ \Delta k \to 0}} \sum_{m=-\infty}^{\infty} \frac{F_{k_m}}{\Delta k} e^{ik_m x} \Delta k = \int_{-\infty}^{\infty} \left(\lim_{\substack{l\to\infty, \\ \Delta k \to 0}} \frac{F_{k_m}}{\Delta k} \right) e^{ikx} \, dk = \int_{-\infty}^{\infty} \hat{F}(k) e^{ikx} \, dk,$$

$$\hat{F}(k) = \lim_{\substack{l\to\infty, \\ \Delta k \to 0}} \frac{F_{k_m}}{\Delta k} = \lim_{\substack{l\to\infty, \\ \Delta k \to 0}} \frac{l}{\pi} \frac{1}{2l} \int_{-l}^{l} f(x) e^{-ik_m x} \, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-ikx} \, dx$$

$$f(x) = \int_{-\infty}^{\infty} \hat{F}(k) e^{ikx} \, dk \qquad \text{(Inverse Transform)} \quad (8)$$

$$\hat{F}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-ikx} \, dx \qquad \text{(Forward Transform)} \quad (9)$$

※ Dimensions of $f(x)$ and $\hat{F}(k)$ are different.

$$\left([f(x)] = \left[k\hat{F}(k) \right], \ \left[\hat{F}(k) \right] = [x f(x)] \right)$$

Orthogonality of continuous function

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(m-m')\theta}\,d\theta = \delta_{m,m'} \quad (3)$$

$$\frac{1}{2l}\int_{-l+a}^{l+a} e^{i(k_m-k_{m'})x}\,dx = \delta_{m,m'} \quad (10)$$

In the case of $l \to \infty$

$$\int_{-\infty}^{\infty} e^{i(k-k')x}\,dx$$

becomes ...?

Taking $l \to \infty$ in Eq. (10) $\lim_{l\to\infty}\frac{1}{2l}\int_{-l+a}^{l+a} e^{i(k_m-k_{m'})x}\,dx = \delta_{m,m'}$

Multiplying $2l$ for both sides of Eq. (10)

$$\text{LHS} = \lim_{l\to\infty}\int_{-l+a}^{l+a} e^{i(k_m-k_{m'})x}\,dx = \int_{-\infty}^{\infty} e^{i(k-k')x}\,dx$$

$$\text{RHS} = \lim_{l\to\infty} 2l\,\delta_{m,m'} = \begin{cases} \infty & (k = k') \\ 0 & (k \neq k') \end{cases}$$

(RHS=$\delta(k-k')$??
Can we handle as a symmetric function, although the original is an asymmetric function?
Is the result a positive real value?)

The INVERSE transform of the FORWARD transform must equal the original function.

$$f(x) = \int_{-\infty}^{\infty} F(k) e^{ikx}\, dk \tag{8}$$

$$F(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx \tag{9}$$

After substituting Eq. (8) into RHS of Eq. (9), exchange the order of integrals.

$$F(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} F(k') e^{ik'x}\, dk' \right) e^{-ikx}\, dx = \int_{-\infty}^{\infty} F(k') \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(k'-k)x}\, dx \right) dk'$$

The δ function has a following nature.

$$F(k) = \int_{-\infty}^{\infty} F(k') \delta(k' - k)\, dk'$$

From comparisons of them we can derive the following orthogonality.

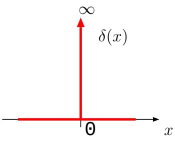
$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(k-k')x}\, dx = \delta(k - k') \tag{11}$$

Dirac's δfunction

Definition of δ func.

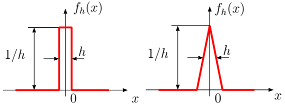
$$\delta(x) = \begin{cases} \infty (x = 0) \\ 0 \ (x \neq 0) \end{cases} \qquad (12)$$

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1 \qquad (13)$$



$$\int_{-\infty}^{\infty} f(x)\delta(x - x')\, dx = f(x') \qquad (14)$$

eg. $\delta(x) = \lim_{h \to 0} f_h(x)$



Dimension of δ func.

$$[\delta(x)] = \left[\frac{1}{x}\right]$$

Natures in the evaluation of integrals (Natures of integral of product with $f(x)$)

$$\delta(-x) = \delta(x), \quad \delta^{(1)}(-x) = -\delta^{(1)}(x), \quad x\delta(x) = 0, \quad \delta(ax) = \frac{1}{|a|}\delta(x), \quad \cdots$$

Various expressions of Fourier Transform

The sufficient condition is INV$\{$FWD$\{f\}\} = f$.

○ Arbitrary polarity of index in exponential function

$$\begin{cases} f(x) = \int_{-\infty}^{\infty} F(k) e^{\pm ikx}\, dk \\ F(k) = \dfrac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{\mp ikx}\, dx \end{cases}$$

○ Arbitrariness of factor

$$\begin{cases} f(x) = A \int_{-\infty}^{\infty} F(k) e^{+ikx}\, dk \\ F(k) = A' \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx \end{cases}$$

$$AA' = \frac{1}{2\pi}$$ (Sufficient condition)

A	A'	Nature
$\frac{1}{\sqrt{2\pi}}$	$\frac{1}{\sqrt{2\pi}}$	symmetrical form
$\frac{1}{2\pi}$	1	When $k = 2\pi\bar{k}$, both factors equals to 1.
1	$\frac{1}{2\pi}$	