Image Processing

S. Tomioka

tom@qe.eng.hokudai.ac.jp

1. Fourier Transform (Series Expansion)

time t [s]	$\Longleftrightarrow$	frequency ω [rad/s]
position $\boldsymbol{r}$ [m]	$\Longleftrightarrow$	wave number $\boldsymbol{k}$ [rad/m]

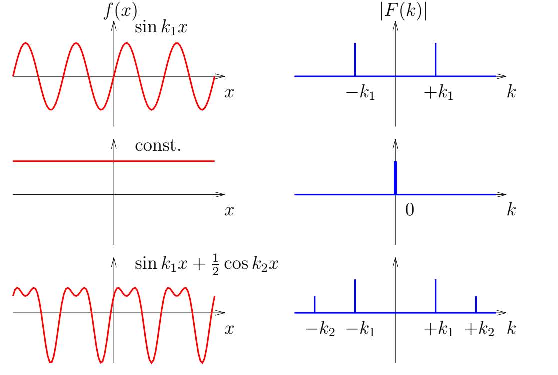
◎ Advantages of Fourier Transform

- understanding of data characteristic (period)
- understanding data propagation mechanism
  (Convolution $\leftrightarrow$ Product in Fourier space)
- Filtering (reduction of noise, enhancement of certain characteristic)
- simple mathematical operation

  ⋮

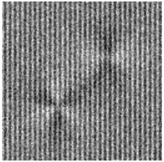
○ Data characteristic

eg.

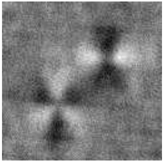


We can understand periodicity of $f(x)$ from spectrum $|F(k)|$.

○ example of filtering



$\downarrow$



1.1 Complex Fourier Series

Complete Orthogonal System

When a complex function $f(\theta); \theta \in [-\pi, +\pi]$ satisfies

- $\int_a^b |f(\theta)| \, d\theta < \infty$
- $f(-\pi) = f(+\pi)$,

$f(\theta)$ is expressed by a series of $e^{im\theta}$.

$$f(\theta) = \sum_{m=-\infty}^{\infty} F_m e^{im\theta} \quad (1)$$

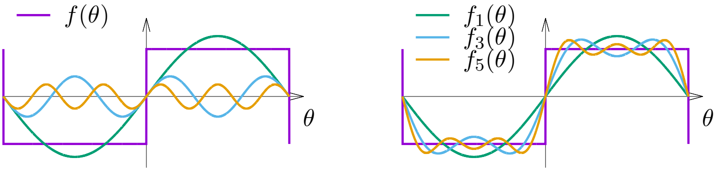
$$F_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-im\theta} \, d\theta \quad (2)$$

$$(f \in \mathbb{C},\ \theta \in \mathbb{R},\ m \in \mathbb{Z}(\text{Integer Numbers}))$$

Completeness

$$f(\theta) = \sum_{m=-\infty}^{\infty} F_m e^{im\theta} \qquad \text{(Completeness)} \tag{1}$$

Proof: Show $\lim_{N\to\infty} f_N(\theta) - f(\theta) = 0$. (detail is omitted)



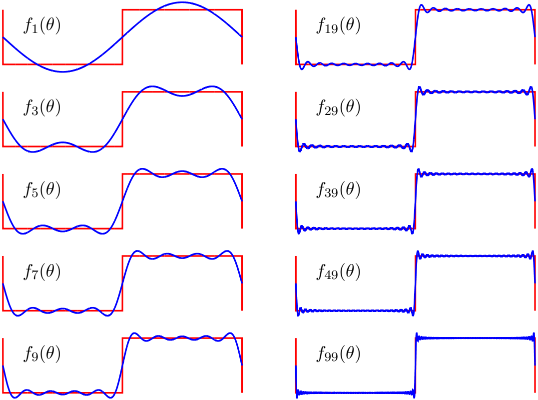
$f_N(x)$: Approximated function by truncated finite terms
$$\left(f_N(\theta) = \sum_{m=-N}^{N} F_m e^{im\theta} \right)$$

※ Discontinuities of $f(\theta)$ between the domain, or the inconsistency at both ends ($f(-\pi) \neq f(+\pi)$) are acceptable.
In these cases, the value at the discontinuous point is considered as the average. (Dirichlet's theorem)

Gibbs phenomenon

- The truncated function with finite terms $f_N(\theta)$ has error with dumping oscillation around discontinuous point.

$$f_N(\theta) = \sum_{m=-N}^{N} F_m e^{im\theta}$$



Orthogonality

$$f(\theta) = \sum_{m=-\infty}^{\infty} F_m e^{im\theta} \tag{1}$$

$$F_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-im\theta}\, d\theta \tag{2}$$

We can prove Eq. (2) using the following orthogonal property.

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(m-m')\theta}\, d\theta = \delta_{m,m'} = \begin{cases} 1 & (m = m') \\ 0 & (m \neq m') \end{cases} \qquad \text{(Orthogonality)} \quad (3)$$

Scaling of domain($\theta \in [-\pi, +\pi] \ \to x \in [-l, l]$)

$$f(\theta) = \sum_{m=-\infty}^{\infty} F_m e^{im\theta} \qquad (1)$$

$$F_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-im\theta} \, d\theta \qquad (2)$$

$$\Rightarrow \left(\ x = \frac{l}{\pi}\theta, \qquad k_m = m\frac{\pi}{l} \in \mathbb{R} \ \right)$$

$$\Downarrow$$

$$f(x) = \sum_{m=-\infty}^{\infty} F_{k_m} e^{ik_m x} \qquad (4)$$

$$F_{k_m} = \frac{1}{2l} \int_{-l}^{l} f(x) e^{-ik_m x} \, dx \qquad (5)$$

※ Both dimensions of $f(x)$ and F_{k_m} are same.

(* Arguments of elementary functions has no dimensions. (Exception: x of $\log(x)$)
* The value of elementary function has also no dimensions.)

Shifting origin($x \in [-l, +l] \ \to x' \in [-l+a, l+a]$)

$$f(x) = \sum_{m=-\infty}^{\infty} F_{k_m} e^{ik_m x} \tag{4}$$

$$F_{k_m} = \frac{1}{2l} \int_{-l}^{l} f(x) e^{-ik_m x} \, dx \tag{5}$$

$\xrightarrow{x \to x+a}$

$$f(x+a) = \sum_{m=-\infty}^{\infty} F_{k_m} e^{ik_m (x+a)}$$

$$F_{k_m} = \frac{1}{2l} \int_{-l}^{l} f(x+a) e^{-ik_m (x+a)} \, dx$$

$\Downarrow \quad (x' = x + a)$

$$f(x') = \sum_{m=-\infty}^{\infty} F_{k_m} e^{ik_m x'} \tag{6}$$

$$F_{k_m} = \frac{1}{2l} \int_{-l+a}^{l+a} f(x') e^{-ik_m x'} \, dx' \tag{7}$$

Only the change of integration range

Fourier Transform($l \to \infty$)

$$f(x) = \sum_{m=-\infty}^{\infty} F_{k_m} e^{ik_m x}, \quad F_{k_m} = \frac{1}{2l}\int_{-l}^{l} f(x) e^{-ik_m x}\, dx \qquad \left(k_m = m\frac{\pi}{l} = m\Delta k\right)$$

If $l \to \infty$, then $\Delta k \to 0$, discrete $k_m \to$ continuous k, , $\sum_{m=-\infty}^{\infty} \cdots \Delta k \to \int_{-\infty}^{\infty} \cdots\, dk$

$$f(x) = \lim_{\substack{l\to\infty,\\ \Delta k\to 0}} \sum_{m=-\infty}^{\infty} \frac{F_{k_m}}{\Delta k} e^{ik_m x}\Delta k = \int_{-\infty}^{\infty} \left(\lim_{\substack{l\to\infty,\\ \Delta k\to 0}} \frac{F_{k_m}}{\Delta k}\right) e^{ikx}\, dk = \int_{-\infty}^{\infty} \hat{F}(k) e^{ikx}\, dk,$$

$$\hat{F}(k) = \lim_{\substack{l\to\infty,\\ \Delta k\to 0}} \frac{F_{k_m}}{\Delta k} = \lim_{\substack{l\to\infty,\\ \Delta k\to 0}} \frac{l}{\pi}\frac{1}{2l}\int_{-l}^{l} f(x) e^{-ik_m x}\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx$$

$$f(x) = \int_{-\infty}^{\infty} \hat{F}(k) e^{ikx}\, dk \qquad \text{(Inverse Transform)} \quad (8)$$

$$\hat{F}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx \qquad \text{(Forward Transform)} \quad (9)$$

※ Dimensions of $f(x)$ and $\hat{F}(k)$ are different.

$$\left([f(x)] = \left[k\hat{F}(k)\right],\ \left[\hat{F}(k)\right] = [x f(x)]\right)$$

Orthogonality of continuous function

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(m-m')\theta}\,d\theta = \delta_{m,m'} \quad (3)$$

$$\frac{1}{2l}\int_{-l+a}^{l+a} e^{i(k_m-k_{m'})x}\,dx = \delta_{m,m'} \quad (10)$$

In the case of $l \to \infty$

$$\int_{-\infty}^{\infty} e^{i(k-k')x}\,dx$$

becomes ...?

Taking $l \to \infty$ in Eq. (10) $\lim_{l\to\infty}\frac{1}{2l}\int_{-l+a}^{l+a} e^{i(k_m-k_{m'})x}\,dx = \delta_{m,m'}$

Multiplying $2l$ for both sides of Eq. (10)

$$\text{LHS} = \lim_{l\to\infty}\int_{-l+a}^{l+a} e^{i(k_m-k_{m'})x}\,dx = \int_{-\infty}^{\infty} e^{i(k-k')x}\,dx$$

$$\text{RHS} = \lim_{l\to\infty} 2l\,\delta_{m,m'} = \begin{cases} \infty & (k=k') \\ 0 & (k\neq k') \end{cases}$$

(RHS=$\delta(k-k')$??
Can we handle as a symmetric function, although the original is an asymmetric function?
Is the result a positive real value?)

The INVERSE transform of the FORWARD transform must equal the original function.

$$f(x) = \int_{-\infty}^{\infty} F(k)e^{ikx}\,dk \quad (8)$$

$$F(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx \quad (9)$$

After substituting Eq. (8) into RHS of Eq. (9), exchange the order of integrals.

$$F(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} F(k')e^{ik'x}\,dk'\right)e^{-ikx}\,dx = \int_{-\infty}^{\infty} F(k')\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(k'-k)x}\,dx\right)dk'$$

The δ function has a following nature.

$$F(k) = \int_{-\infty}^{\infty} F(k')\delta(k'-k)\,dk'$$

From comparisons of them we can derive the following orthogonality.

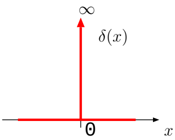
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(k-k')x}\,dx = \delta(k-k') \quad (11)$$

Dirac's δfunction

Definition of δ func.

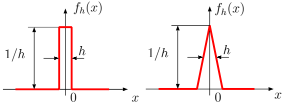
$$\delta(x) = \begin{cases} \infty (x = 0) \\ 0 \ (x \neq 0) \end{cases} \qquad (12)$$

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1 \qquad (13)$$



$$\int_{-\infty}^{\infty} f(x)\delta(x - x')\, dx = f(x') \qquad (14)$$

eg. $\delta(x) = \lim_{h \to 0} f_h(x)$



Dimension of δ func.

$$[\delta(x)] = \left[\frac{1}{x}\right]$$

Natures in the evaluation of integrals (Natures of integral of product with $f(x)$)

$$\delta(-x) = \delta(x), \quad \delta^{(1)}(-x) = -\delta^{(1)}(x), \quad x\delta(x) = 0, \quad \delta(ax) = \frac{1}{|a|}\delta(x), \quad \cdots$$

Various expressions of Fourier Transform

The sufficient condition is INV$\{$FWD$\{f\}\} = f$.

○ Arbitrary polarity of index in exponential function

$$\begin{cases} f(x) = \int_{-\infty}^{\infty} F(k) e^{\pm ikx}\, dk \\ F(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{\mp ikx}\, dx \end{cases}$$

○ Arbitrariness of factor

$$\begin{cases} f(x) = A \int_{-\infty}^{\infty} F(k) e^{+ikx}\, dk \\ F(k) = A' \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx \end{cases}$$

$$AA' = \frac{1}{2\pi}$$ (Sufficient condition)

A	A'	Nature
$\frac{1}{\sqrt{2\pi}}$	$\frac{1}{\sqrt{2\pi}}$	symmetrical form
$\frac{1}{2\pi}$	1	When $k = 2\pi\bar{k}$, both factors equals to 1.
1	$\frac{1}{2\pi}$	

1.2 Discrete Fourier Transform (DFT)

$$f'(x) = \sum_{m=-\infty}^{\infty} F'_{k_m} e^{ik_m x} \tag{4}$$

$$F'_{k_m} = \frac{1}{L} \int_0^L f'(x) e^{-ik_m x}\, dx \tag{5}$$

$$k_m = m\frac{2\pi}{L} \tag{15}$$

(In exact, "Discrete Fourier Series Expansion".)

Consider the case when $f(x)$ is sampled with even intervals.

$$(n = 0, 1, \cdots, N-1)$$

$$f_n \equiv f'(x_n), \quad x_n \equiv n\Delta x$$

(Represent integral by summation:

$$\int_0^L \cdots\, dx \simeq \sum_{n=0}^{N-1} \cdots\, \Delta x$$

$$(L = N\Delta x)$$

$$\to F'_{k_m} \simeq \frac{1}{L} \sum_{n=0}^{N-1} f_n e^{-ik_m x_n} \Delta x$$
)

$$f_n = \sum_{m=0}^{N-1} F_m e^{+i\frac{2\pi nm}{N}} \tag{16}$$

$$F_m = \frac{1}{N} \sum_{n=0}^{N-1} f_n e^{-i\frac{2\pi nm}{N}} \tag{17}$$

Periodicity of F_m and f_n

$$F_{m'} = \frac{1}{N}\sum_{n=0}^{N-1} f_n e^{-i\frac{2\pi n m'}{N}} \qquad (17)$$

Replacing $m' = m + N$

$$\begin{aligned} F_{m+N} &= \frac{1}{N}\sum_{n=0}^{N-1} f_n\, e^{-i\frac{2\pi n(m+N)}{N}} \\ &= \frac{1}{N}\sum_{n=0}^{N-1} f_n\, e^{-i\frac{2\pi n m}{N}} \underbrace{e^{-i2\pi n}}_{=1} = F_m \end{aligned}$$

F_m has a periodicity.

$$F_{m\pm N} = F_m \qquad (18)$$

$$f_{n'} = \sum_{m=0}^{N-1} F_m e^{+i\frac{2\pi n' m}{N}} \qquad (16)$$

F_m also has a periodicity.

$$f_{n\pm N} = f_n \qquad (19)$$

We can choose any set of N points for m in F_m or n in f_n, if the point is not a periodic points of others.
However, we must consider the sampling theorem for interpolations.

Sampling Theorem

Signal consist of single sinusoidal function:

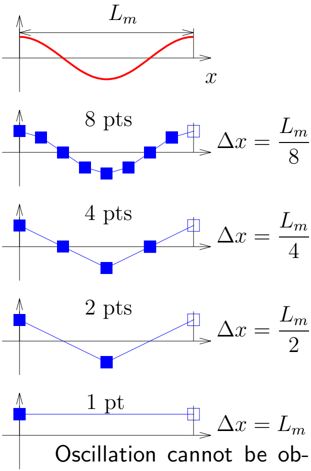
$$f_m(x) = F_m e^{ik_m x} \quad (k_m = \tfrac{2\pi}{L_m})$$

In order to observe oscillation with a period L_m, two points are needed within L_m.

Sampling Theorem

$$\Delta x \leq \frac{L_m}{2}$$

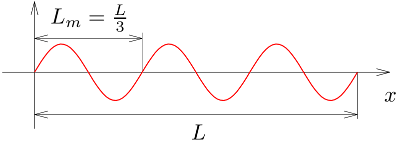
$$k_m \leq \frac{\pi}{\Delta x} \quad \text{(Nyquist Freq.)} \tag{20}$$



Oscillation cannot be observed, with $\Delta x = L_m$.

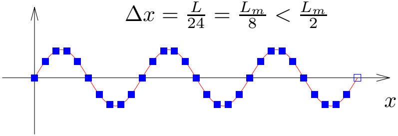
Sampling Theorem and Aliasing

Case of $m = 3$:



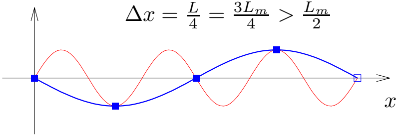
Fine sampling $(N = 24)$

$\Delta x = \frac{L}{24} = \frac{L_m}{8} < \frac{L_m}{2}$



Coarse sampling $(N = 4)$

$\Delta x = \frac{L}{4} = \frac{3L_m}{4} > \frac{L_m}{2}$



$\Delta x > \dfrac{L}{2} \rightarrow$ Different sinusoidal function is observed. $\rightarrow$ Aliasing

$$\left(\begin{array}{l} \text{Freq. of true signal:} \\ \quad k_m = m\dfrac{2\pi}{L} \\ \text{Freq. of spurious signal:} \\ \quad k'_m = (m-N)\dfrac{2\pi}{L} = k_{m-N} \end{array} \right)$$

If the sampling interval is Δx, the signal with $k_m > \dfrac{\pi}{\Delta x}$ cannot be observed.

$\rightarrow$Sampling Theorem

In this condition, the aliased signal with the following frequency is observed.

$$\frac{-\pi}{\Delta x} \leq k_{m'} = k_{m-N} \leq \frac{\pi}{\Delta x}$$

$$(|m-N| \leq N/2)$$

Relation of domains and intervals between real and Fourier spaces

$$\begin{cases} L = x_{\max} - x_{\min} = N\Delta x & (f_n = f_{n\pm N}) \\ K = k_{\max} - k_{\min} = N\Delta k & (F_m = F_{m\pm N}) \end{cases}$$

- Sampling Theorem: $|k| < \dfrac{\pi}{\Delta x}$
- Symmetry of k $(k_{\min} = -k_{\max})$

$$\left(k_{\max} = -k_{\min} = \frac{\pi}{\Delta x}\right)$$

$$K = k_{\max} - k_{\min} = \frac{2\pi}{\Delta x}$$

$$\Delta k = \frac{2\pi}{x_{\max} - x_{\min}} = \frac{2\pi}{L}$$

(Real space)		(Fourier space)
Δx	Reciprocal $\Longleftrightarrow$ (Prod.$=2\pi$)	K
$\times N \Downarrow$		$\Uparrow \times N$
L	Reciprocal $\Longleftrightarrow$ (Prod.$=2\pi$)	Δk

Summary of DFT

FT of signal with discrete sampling (num. of samp.$=N$)

$$f_n = \sum_{\substack{m \\ (x_n = n\Delta x)}} F_m e^{ik_m x_n} \quad \underset{\text{INV}}{\overset{\text{FWD}}{\rightleftarrows}} \quad F_m = \frac{1}{N} \sum_{\substack{n \\ (k_m = m\Delta k)}} f_n e^{-ik_m x_n}$$

- Suffixes n and m of f_n and F_m have periodicity with the period N.
- In the computation of f_n and F_m, arbitrary set of m and n has same result because of this periodicity. (eg. $n, m = \{0, \cdots, N-1\}$, or $n, m = \{\lfloor -N/2 \rfloor, \cdots, \lfloor N/2 - 1 \rfloor\}$)
- However, if m of F_m is $|m| > N/2$, m should be shifted into $|m| \leq N/2$ to satisfy sampling theorem.
- Interpolation

  Once F_m is obtained, we can evaluate $f(x|x \neq n\Delta x)$ by inverse transform. In this case, m must satisfy the sampling theorem. Otherwise, the interpolated function shows an aliasing function.

Techniques to compute DFT

$$F_m = \frac{1}{N}\sum_{n=0}^{N-1} f_n e^{-i\frac{2\pi mn}{N}} \qquad (17)$$

- In a simple computation, most of computational cost is consumed to evaluate exponential function $e^{-i\frac{2\pi mn}{N}}$.
- To evaluate all of F_m, N^2 times evaluations are needed.
- Argument of exponential function

$$\frac{mn}{N} = \underbrace{\left\lfloor \frac{mn}{N} \right\rfloor}_{\text{Integer}} + \frac{1}{N}\underbrace{(mn)\%N}_{\text{Fraction}}$$

$$mn\%N \in \{0, 1, \cdots, N-1\}$$

- Since $e^{i2l\pi} = 1$ for $l \in \mathbb{Z}$, $e^{-i\frac{2\pi mn}{N}} = e^{-i\frac{2\pi (mn)\%N}{N}}$.

If we evaluate $W^p = e^{-\frac{i2\pi}{N}p}$ for $p \in \{0, 1, \cdots, N\}$ at the first, the number of times for exponential evaluations is only N.

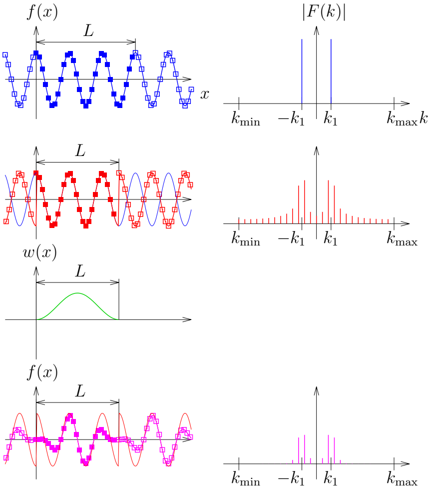
(When we use this table, the time to compute multiplication governs the computational time.
The scheme to reduce the num. of times for multiplication is called FFT.)

Window function

Sinusoidal signal: $f_{x_n} = \cos k_1 x_n$

- Case of $f_{x_0} = f_{x_N}$
  $|F_{k_m}| = \frac{1}{2}(\delta_{m,n} + \delta_{-m,n})$
- Case of $f_{x_0} \neq f_{x_N}$
  $|F(k)|$ spreads around k_1.

  - Reason:
    Assumed periodicity.
  - Solution: Multiply window function $w(x)$ so that the ends becomes continuous.
    $$f'(x) = w(x) f(x)$$
  - example of the window function:
    $$w(x) = \frac{1}{2}\left(1 + \cos\frac{2\pi(x - x_c)}{L}\right)$$
    $$(x \in [x_c - L/2, x_c + L/2])$$



Two-dimensional DFT

$$F_{m_x,m_y} = \frac{1}{N_x N_y} \sum_{n_x=0}^{N_x-1} \sum_{n_y=0}^{N_y-1} f_{n_x,n_y} e^{-i\left(\frac{2\pi m_x n_x}{N_x} + \frac{2\pi m_y n_y}{N_y}\right)} \tag{21}$$

To evaluate all F_{m_x,m_y} $(N_x N_y)^2$ times multiplication are needed. (4 loops)

(Exchange the order of operators)

$$F_{m_x,m_y} = \frac{1}{N_x} \sum_{n_x=0}^{N_x-1} \underbrace{\left\{ \frac{1}{N_y} \sum_{n_y=0}^{N_y-1} f_{n_x,n_y} e^{-i\frac{2\pi m_y n_y}{N_y}} \right\}}_{=G_{n_x,m_y}} e^{-i\frac{2\pi m_x n_x}{N_x}}$$

$$G_{n_x,m_y} = \frac{1}{N_y} \sum_{n_y=0}^{N_y-1} f_{n_x,n_y} e^{-i\frac{2\pi m_y n_y}{N_y}}, \qquad F_{m_x,m_y} = \frac{1}{N_x} \sum_{n_x=0}^{N_x-1} G_{n_x,m_y} e^{-i\frac{2\pi m_x n_x}{N_x}}$$

$\left\{ \begin{matrix} G_{n_x,m_y} \\ F_{m_x,m_y} \end{matrix} \right\}$ is the FT of $\left\{ \begin{matrix} f_{n_x,n_y} \\ G_{n_x,m_y} \end{matrix} \right\}$ with $\left\{ \begin{matrix} m_y \\ m_x \end{matrix} \right\}$. The num. of multi. for all is $\left\{ \begin{matrix} N_x N_y^2 \\ N_x^2 N_y \end{matrix} \right\}$.
Total num. of operation is reduced to $N_x N_y^2 + N_x^2 N_y$ times. (3 loops for each step. 2 steps.)

1.3 Fast Fourier Transform (FFT)

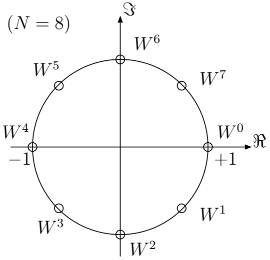
$$\hat{F}_m = \frac{1}{N}\sum_{n=0}^{N-1} f_n e^{-i\frac{2\pi}{N}mn} \tag{17}$$

$$F_m \triangleq \sum_{n=0}^{N-1} f_n \underbrace{e^{-i\frac{2\pi}{N}mn}}_{W^{mn}} \quad \left(\hat{F}_m = \tfrac{1}{N}F_m\right)$$

$$W = e^{-i\frac{2\pi}{N}}$$

$$F_m = \sum_{n=0}^{N-1} f_n W^{mn}$$

(FFT: Fast Fourier Transform)



In the case of $N = 2^p$ $(p \in \mathbb{N})$, $W^{mn \pm N/2} = -W^{mn}$ is satisfied.
By using this relation, the number of times of multiplications can be reduced.

Divide n of f_n, $n \in \{0, \cdots, N-1\}$, into two groups which are the even group and the odd group.

$$f^e_{n'} = f_{2n'}, \quad f^o_{n'} = f_{2n'+1} \qquad (n' \in \{0, 1, ..., N/2-1\})$$

$$\left(\begin{aligned} F_m &= \sum_{n=0}^{N-1} f_n W^{mn} = \sum_{n'=0}^{N/2-1} f^e_{n'} W^{2mn'} + \sum_{n'=0}^{N/2-1} f^o_{n'} W^{m(2n'+1)} \\ &= \underbrace{\sum_{n=0}^{N/2-1} f^e_n W^{2mn}}_{\text{DFT of } f^e_n \equiv F^e_m (N/2\text{points})} + W^m \underbrace{\sum_{n=0}^{N/2-1} f^o_n W^{2mn}}_{\text{DFT of } f^o_n \equiv F^o_m (N/2\text{points})} = F^e_m + W^m F^o_m \end{aligned} \right)$$

Replace $(m \to N/2 + m)$

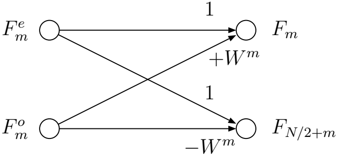
$$\left(\begin{aligned} F_{\frac{N}{2}+m} &= \sum_{n=0}^{N/2-1} f^e_n \overbrace{W^{2(N/2+m)n}}^{W^{2mn}} + \overbrace{W^{N/2+m}}^{-W^m} \sum_{n=0}^{N/2-1} f^o_n \overbrace{W^{2(N/2+m)n}}^{W^{2mn}} \\ &= \underbrace{\sum_{n=0}^{N/2-1} f^e_n W^{2mn}}_{\text{DFT of } f^e_n \equiv F^e_m (N/2\text{points})} - W^m \underbrace{\sum_{n=0}^{N/2-1} f^o_n W^{2mn}}_{\text{DFT of } f^o_n \equiv F^o_m (N/2\text{points})} = F^e_m - W^m F^o_m \end{aligned} \right)$$

Butterfly operation

$$F_m^{\left\{ e \atop o \right\}} = \sum_{n=0}^{N/2-1} f_n^{\left\{ e \atop o \right\}} W^{2nm} \quad (22)$$

$$\begin{cases} F_m = F_m^e + W^m F_m^o \\ F_{N/2+m} = F_m^e - W^m F_m^o \end{cases} \quad (23)$$

$$(m \in \{0, \cdots, N/2-1\})$$

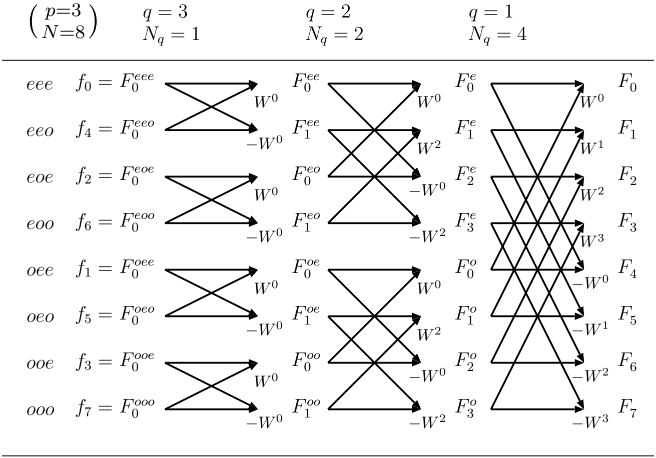


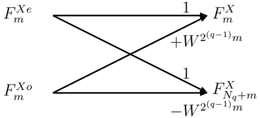
Butterfly operation

- If we know both F_m^e and F_m^o, we can evaluate both F_m and $F_{N/2+m}$.
- In order to evaluate F_m and $F_{N/2+m}$, $N/2$ times of multiplications for each are needed. The sum of them are N times.
- In order to obtain F_m^e and F_m^o, $\cdots$.

Divide each of F_m^e and F_m^o into two groups. Furthermore, repeat dividing.

- 1st time ($N_1 = N/2$) $F_{\left\{ \substack{m \\ m+N_1} \right\}} = F_m^e \pm W^m F_m^o$
- 2nd time ($N_2 = N/2^2$) $F^x_{\left\{ \substack{m \\ m+N_2} \right\}} = F_m^{xe} \pm W^{2m} F_m^{xo} \quad (x \in \{e, o\})$
- 3rd time ($N_3 = N/2^3$) $F^{x_1 x_2}_{\left\{ \substack{m \\ m+N_3} \right\}} = F_m^{x_1 x_2 e} \pm W^{2^2 m} F_m^{x_1 x_2 o}$ $(x_0, x_1 \in \{e, o\})$
- q-th time ($N_q = N/2^q$) $F^{\boldsymbol{X}}_{\left\{ \substack{m \\ m+N_q} \right\}} = F_m^{\boldsymbol{X} e} \pm W^{2^{(q-1)} m} F_m^{\boldsymbol{X} o}$ $(\boldsymbol{X} = (x_0\ x_1\ \cdots x_{q-2}), x_i \in \{e, o\})$





$X = (x_0, ..., x_{p-q-1})$

$x_i \in \{e, o\}$

Times of multiplications:

N/stage $\times$ p stage $= N \log_2 N$

Bit-reversal scheme

Relation between $F_m^{\boldsymbol{X}}$ and f_n (e.g. $N = 8, p = 3$)

F_m^{eo} (Pick up the even group in the first. After that pick up the odd group.)								
0 (0)	1 (1)	2 (2)	3 (3)	4 (4)	5 (5)	6 (6)	7 (7)	pick up the even group (e)
0 (0)		2 (1)		4 (2)		6 (3)		pick up the even group (o)
		2 (0)				6 (1)		$\rightarrow F_m^{eo}$: FT of two points for f_2 and f_6.
F_0^{eoo} (Pick up groups in order of even, odd, and odd.)								
						6		$\rightarrow F_0^{eoo}$:FT of the single point for f_6.

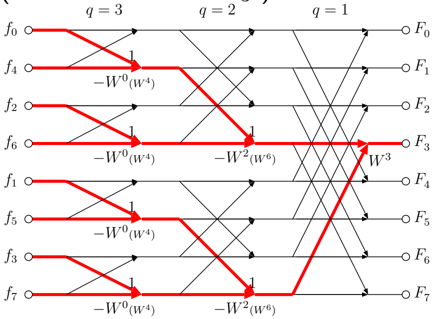
Bit-reversal scheme

(a)	parity	eee	eeo	eoe	eoo	oee	oeo	ooe	ooo
					$e \rightarrow 0,\ o \rightarrow 1$				
(b)	binary (decimal)	000 (0)	001 (1)	010 (2)	011 (3)	100 (4)	101 (5)	110 (6)	111 (7)
(c)	reversal (decimal)	000 (0)	100 (4)	010 (2)	110 (6)	001 (1)	101 (5)	011 (3)	111 (7)

We can obtain the list of n for f_n by using bit reverse.

Example of FFT operation

(Path to evaluate F_3.)



Product along path ($W^8 = 1, -1 = W^4$)					check		
beg.	St. 1	St. 2	St. 3	Product ○	n	$3n$	$3n\%8$ ○
f_0	1	1	1	$1 = W^0$	0	0	0
f_4	$-W^0$	1	1	$-W^0 = W^4$	4	12	4
f_2	1	$-W^2$	1	$-W^2 = W^6$	2	6	6
f_6	$-W^0$	$-W^2$	1	W^2	6	18	2
f_1	1	1	W^3	W^3	1	3	3
f_5	$-W^0$	1	W^3	$-W^3 = W^7$	5	15	7
f_3	1	$-W^2$	W^3	$-W^5 = W^1$	3	9	1
f_7	$-W^0$	$-W^2$	W^3	W^5	7	21	5

We can confirm that the columns with "○" are same.

$$F_m = \sum_{n=0}^{N-1} f_n W^{nm}$$

To apply FFT

- The FFT can be applied only when $N = 2^p$.
- In the case of $N \neq 2^p$ $(2^{p-1} < N < 2^p)$:
  - Remove data with less information around ends so that $N' = 2^{p-1}$.
  - Add data $f_n = f^\dagger$ for $n \in N, ..., 2^p$ so that $N' = 2^p$. (padding) $\left(\text{Padding data } f^\dagger\text{: } \bar{f}\text{(ave.), } 0\text{, or } \frac{f_0 + f_{N-1}}{2}\right)$
  - A window function after adding or removing is multiplied, if necessary.

Summary of FFT

- When $N = 2^p$, the times of multiplications can be reduced by using butterfly operations.
- Comparison of the times of multiplications

$$\begin{aligned} N_{\mathsf{Mul}}(\mathsf{FFT}) &= N \log_2 N \\ N_{\mathsf{Mul}}(\mathsf{DFT}) &= N^2 \end{aligned} \quad \rightarrow \quad N_{\mathsf{Mul}}(\mathsf{FFT}) \ll N_{\mathsf{Mul}}(\mathsf{DFT})$$

  FFT is more effective with increasing N.

  e.g.

N	32	1024	32768
DFT	~ 1000	$\sim 10^6$	$\sim 10^9$
FFT	160	$\sim 10^4$	$\sim 5 \times 10^5$

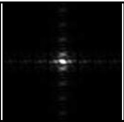
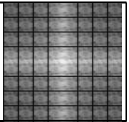
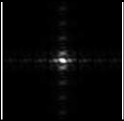
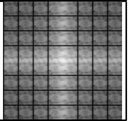
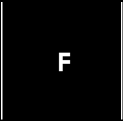
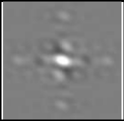
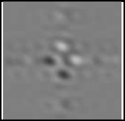
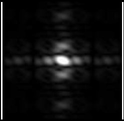
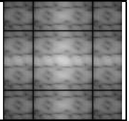
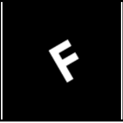
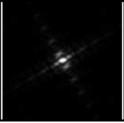
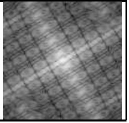
- In the case of a two-dimensional image ($N_x \times N_y$), FFT can be applied only for the most inner loop.

$$N_{\mathsf{Mul}}(\mathsf{FFT}) = N_x N_y (\log_2 N_x + \log_2 N_y)$$

$$N_{\mathsf{Mul}}(\mathsf{DFT}) = N_x N_y (N_x + N_y)$$

1.4 Characteristices of Fourier Transform

Examples of Fourier Transform (Origin : center)

Ope.	$f(\boldsymbol{r}) \in \mathbb{R}$	$\Re\{F(\boldsymbol{k})\}$	$\Im\{F(\boldsymbol{k})\}$	$\lvert F(\boldsymbol{k})\rvert$	$\log\lvert F(\boldsymbol{k})\rvert$
Orig.					
Shift					
Scale down					
Rotation					

Symmetry

$$f(\boldsymbol{r}) = a(\boldsymbol{r}) + ib(\boldsymbol{r}) \quad (a, b \in \mathbb{R},\ f \in \mathbb{C})$$
$$F(\boldsymbol{k}) = A(\boldsymbol{k}) + iB(\boldsymbol{k}) \quad (A, B, F \in \mathbb{C})$$
$$= A + iB \qquad \text{(omit } (\boldsymbol{k})\text{)}$$
$$F(\boldsymbol{k}) \triangleq \int f(\boldsymbol{r}) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\, d\boldsymbol{r}$$
$$\left(\begin{array}{l} \begin{Bmatrix} A(\boldsymbol{k}) \\ B(\boldsymbol{k}) \end{Bmatrix} = \int \begin{Bmatrix} a(\boldsymbol{r}) \\ b(\boldsymbol{r}) \end{Bmatrix} e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\, d\boldsymbol{r} \\ \begin{Bmatrix} A(-\boldsymbol{k}) \\ B(-\boldsymbol{k}) \end{Bmatrix} = \int \begin{Bmatrix} a(\boldsymbol{r}) \\ b(\boldsymbol{r}) \end{Bmatrix} e^{+i\boldsymbol{k}\cdot\boldsymbol{r}}\, d\boldsymbol{r} \end{array} \right)$$

$$\begin{Bmatrix} A(-\boldsymbol{k}) \\ B(-\boldsymbol{k}) \end{Bmatrix} = \begin{Bmatrix} A^*(\boldsymbol{k}) \\ B^*(\boldsymbol{k}) \end{Bmatrix} = \begin{Bmatrix} A^* \\ B^* \end{Bmatrix}$$
$$F^*(\boldsymbol{k}) = A^* - iB^*$$
$$F(-\boldsymbol{k}) = A^* + iB^* \neq F^*(\boldsymbol{k})$$
$$|F(\boldsymbol{k})|^2 = (A + iB)^*(A + iB)$$
$$= (|A|^2 + |B|^2) + i(A^*B - AB^*)$$
$$|F(-\boldsymbol{k})|^2 = (A^* + iB^*)^*(A^* + iB^*)$$
$$= (|A|^2 + |B|^2) - i(A^*B - AB^*)$$
$$\neq |F(\boldsymbol{k})|^2$$

	$\Re\{F(\boldsymbol{k})\}$	$\Im\{F(\boldsymbol{k})\}$	$\lvert F(\boldsymbol{k})\rvert$
$f(\boldsymbol{r}) = a(\boldsymbol{r})$(Real)	sym.	anti-sym.	sym.
$f(\boldsymbol{r}) = ib(\boldsymbol{r})$(Pure imag.)	anti-sym.	sym.	sym.
$f(\boldsymbol{r}) = a(\boldsymbol{r}) + ib(\boldsymbol{r})$(Complex)	non-sym.	non-sym.	non-sym.

Coordinate transformation

	$\Re\{F(\boldsymbol{k})\}$	$\Im\{F(\boldsymbol{k})\}$	$\lvert F(\boldsymbol{k})\rvert$
Translation			Identical
Scale down	Scale up	Scale up	Scale up
Rotation	Rotation	Rotation	Rotation

Translation

$$f_1(\boldsymbol{r}) = f_0(\boldsymbol{r}+\boldsymbol{\Delta})$$

$$F_1(\boldsymbol{k}) = e^{+i\boldsymbol{k}\cdot\boldsymbol{\Delta}} F_0(\boldsymbol{k})$$

$$\left(\begin{array}{l} (\boldsymbol{r}' = \boldsymbol{r}+\boldsymbol{\Delta}) \\ F_1(\boldsymbol{k}) = \int f_1(\boldsymbol{r}) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\, d\boldsymbol{r} \\ \quad = \int f_0(\boldsymbol{r}+\boldsymbol{\Delta}) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\, d\boldsymbol{r} \\ \quad = \int f_0(\boldsymbol{r}') e^{-i\boldsymbol{k}\cdot(\boldsymbol{r}'-\boldsymbol{\Delta})}\, d\boldsymbol{r}' \\ \quad = e^{+i\boldsymbol{k}\cdot\boldsymbol{\Delta}} \\ \qquad \cdot \int f_0(\boldsymbol{r}') e^{-i\boldsymbol{k}\cdot\boldsymbol{r}'}\, d\boldsymbol{r}' \end{array}\right)$$

Scaling

$$f_1(\boldsymbol{r}) = f_0(\alpha\boldsymbol{r})$$

$$F_1(\boldsymbol{k}) = \frac{1}{\alpha} F_0\left(\frac{\boldsymbol{k}}{\alpha}\right)$$

$$\left(\begin{array}{l} (\boldsymbol{r}' = \alpha\boldsymbol{r}) \\ F_1(\boldsymbol{k}) = \int f_1(\boldsymbol{r}) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\, d\boldsymbol{r} \\ \quad = \int f_0(\alpha\boldsymbol{r}) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\, d\boldsymbol{r} \\ \quad = \int f_0(\boldsymbol{r}') e^{-i\boldsymbol{k}\cdot\frac{\boldsymbol{r}'}{\alpha}}\, \frac{d\boldsymbol{r}'}{\alpha} \\ \quad = \frac{1}{\alpha} \int f_0(\boldsymbol{r}') e^{-i\left(\frac{\boldsymbol{k}}{\alpha}\right)\boldsymbol{r}'\cdot}\, d\boldsymbol{r}' \end{array}\right)$$

Rotation

$$f_1(\boldsymbol{r}) = f_0(\boldsymbol{\Theta}\cdot\boldsymbol{r})$$

$$F_1(\boldsymbol{k}) = F_0\left(\boldsymbol{\Theta}\cdot\boldsymbol{k}\right)$$

$$\left(\begin{array}{l} (\boldsymbol{r}' = \boldsymbol{\Theta}\cdot\boldsymbol{r}) \\ F_1(\boldsymbol{k}) = \int f_1(\boldsymbol{r}) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\, d\boldsymbol{r} \\ \quad = \int f_0(\boldsymbol{\Theta}\boldsymbol{r}) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\, d\boldsymbol{r} \\ \quad = \int f_0(\boldsymbol{r}') e^{-i\boldsymbol{k}\cdot\boldsymbol{\Theta}^{-1}\cdot\boldsymbol{r}'}\, \frac{d\boldsymbol{r}'}{|\boldsymbol{\Theta}|} \\ \quad \left(\begin{array}{l}\boldsymbol{k}\cdot\boldsymbol{\Theta}^{-1} = \boldsymbol{\Theta}\cdot\boldsymbol{k}, \\ |\boldsymbol{\Theta}| = 1\end{array}\right) \\ \quad = \int f_0(\boldsymbol{r}') e^{-i(\boldsymbol{\Theta}\cdot\boldsymbol{k})\cdot\boldsymbol{r}'}\, d\boldsymbol{r}' \end{array}\right)$$

2. Power Spectrum and Correlation function

- What is correlation?
  Relation between two discrete data set $(x_i,\, y_i)$
- What is correlation function?
  Relation between two variates which are represented as continuous function $(x(t),\, y(t))$
- What is power spectrum?
  Measure of Fourier transformed function $X(\omega)$ of a variate $x(t)$.
- Wiener-Khintchine's theorem
  Relation between power spectrum and auto-correlation function

2.1 Definition of Power Spectrum

Fourier Transform :

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega \qquad (1)$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt \qquad (2)$$

Power spectral density

$$S(\omega) = \lim_{T\to\infty}\frac{|X(\omega)|^2}{T} \qquad (3)$$

- Time average of the square of sinusoidal component with frequency ω in the signal.
- No information about phase.
- $S(\omega)\,d\omega$ expresses the power spectrum.
- $|X(\omega)|^2$ is called energy spectrum.
- Dimension analysis:

FT of x	:	$[X] = [x \cdot t]$
Power spectral density	:	$[S] = \left[\frac{\|X\|^2}{T}\right] = [x^2 \cdot t]$
Power spectrum	:	$[S\,d\omega] = \left[\frac{X^2}{t}\frac{1}{t}\right] = [x^2]$
Energy spectral density	:	$[\|X\|^2] = [x^2 t^2]$

2.2 Correlation

Population: Sample f_i $(i \in (1, \cdots, N))$

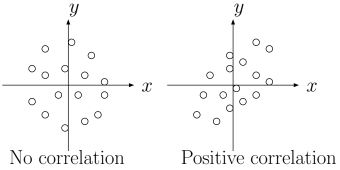
Average $$\mathrm{E}[f] \equiv \frac{1}{N} \sum_{i=1}^{N} f_i \quad (4)$$

Variance $$\sigma_f^2 \equiv \mathrm{E}\left[(f - \mathrm{E}[f])^2\right] \quad (5)$$

Correlation: Index to represent similarity of two variates (x_i, y_i).

$$C = \mathrm{E}\left[x'y'\right], \quad \text{or} \quad r = \frac{\mathrm{E}[x'y']}{\sqrt{\mathrm{E}\left[x'^2\right]\mathrm{E}\left[y'^2\right]}} \quad (6)$$

$$(x_i' = x_i - \mathrm{E}[x], \quad y_i' = y_i - \mathrm{E}[y])$$

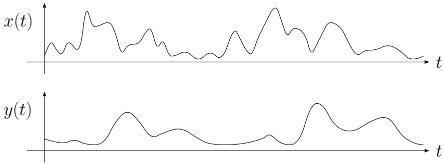


2.3 Correlation Function

In case where x and y are variates with respect to time:

$$\mathrm{E}\left[x\right] \to \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\,dt = \langle x(t)\rangle_t \quad \text{(Time Average)} \tag{7}$$

e.g. $x(t)$: Amount of rainfall,
$y(t)$: Amount of water in a river



- Time delay
- Smoothing of time fluctuation

Cross-correlation function and Auto-correlation function

Cross-correlation function

$$C_{xy}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)y(t+\tau)\,dt = \langle x(t)y(t+\tau)\rangle_t \qquad (8)$$

Even when $y(t) = x(t)$, we can understand the periodicity of $x(t)$.

Auto-correlation function

$$C(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)x(t+\tau)\,dt = \langle x(t)x(t+\tau)\rangle_t \qquad (9)$$

(Normalization)

$$R(\tau) = \frac{C(\tau)}{C(0)} = \frac{\langle x(t)x(t+\tau)\rangle_t}{\langle x(t)^2\rangle_t}, \qquad R(0) = 1$$

$\rightarrow$ auto-correlation coefficient

Periodicity of auto-correlation function

(e.g. 1) $x(t) = A\cos\omega_1 t$

$$C(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}(A\cos\omega_1 t)(A\cos\omega_1(t+\tau))\,dt = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}(A\cos\omega_1 t)(A\cos\omega_1(t+\tau))\,dt$$

$$= A^2\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\left(\cos^2\omega_1 t\cos\omega_1\tau - \cos\omega_1 t\sin\omega_1 t\sin\omega_1\tau\right)\,dt$$

$$= A^2\lim_{T\to\infty}\Bigg[\underbrace{\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\frac{1+\cos 2\omega_1 t}{2}\,dt}_{=\frac{1}{2}}\cos\omega_1\tau - \underbrace{\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\frac{\sin 2\omega_1 t}{2}\,dt}_{=0}\sin\omega_1\tau\Bigg] = \frac{A^2}{2}\cos\omega_1\tau$$

$$\underline{R(\tau) = \cos\omega_1\tau}$$

(e.g. 2) $x(t) = A\sin\omega_1 t = A\cos\left(\omega_1 t - \frac{\pi}{2}\right)$

$$C(\tau) = A^2\lim_{T\to\infty}\left[\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\frac{1+\cos\left(2\left(\omega_1 t - \frac{\pi}{2}\right)\right)}{2}\,dt\cos\omega_1\tau - \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\frac{\sin\left(2\left(\omega_1 t - \frac{\pi}{2}\right)\right)}{2}\,dt\sin\omega_1\tau\right] = \frac{A^2}{2}\cos\omega_1\tau$$

$$\underline{R(\tau) = \cos\omega_1\tau}$$

- Periodicity is found.
- Independent of phase → Independent of the origin of t.

Characteristics of Auto-correlation function

- Independent of the position of origin.
- Even function. $(C(-\tau) = C(\tau))$

$$\left(\begin{aligned} C(-\tau) &= \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)x(t-\tau)\,dt \qquad (t' = t-\tau) \\ &= \lim_{T\to\infty} \int_{-\frac{T}{2}-\tau}^{\frac{T}{2}-\tau} x(t'+\tau)x(t')\,dt' \qquad \left(\frac{T}{2} \gg |\tau|\right) \\ &= \lim_{T\to\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t'+\tau)x(t')\,dt' = C(\tau) \end{aligned} \right)$$

- Maximum at $\tau = 0$.

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \left(x(t) \pm x(t+\tau)\right)^2\,dt \geq 0$$

$$\left(\begin{aligned} \text{LHS} &= \underbrace{\int x^2(t)\,dt}_{C(0)\geq 0} + \underbrace{\int x^2(t+\tau)\,dt}_{C(0)\geq 0} \pm 2 \underbrace{\int x(t)x(t+\tau)\,dt}_{C(\tau)} \\ &= 2(C(0) \pm C(\tau)) \geq 0 = \text{RHS} \\ &\therefore C(0) \geq |C(\tau)| \end{aligned} \right)$$

- The 1st differential of $C(\tau)$

$$\left(\begin{aligned} C'(\tau) &= \frac{dC}{d\tau} = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\underbrace{\frac{\partial x(t+\tau)}{\partial\tau}}_{=x'(t+\tau)}\,dt = \left\langle x(t)x'(t+\tau)\right\rangle_t \\ &\text{(Replacing } t+\tau=\xi \to dt = d\xi, \text{because } \tfrac{\partial(t+\tau)}{\partial\tau}=1) \\ &= \lim_{T\to\infty}\int_{-\frac{T}{2}+\tau}^{\frac{T}{2}+\tau} x(\xi-\tau)x'(\xi)\,d\xi \\ &= \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(\xi-\tau)x'(\xi)\,d\xi = \left\langle x(t-\tau)x'(t)\right\rangle_t \end{aligned}\right)$$

Cross-correlation between $x(t)$ and $x'(t)$.

- The 2nd differential of $C(\tau)$

$$\left(\begin{aligned} C''(\tau) &= \frac{d^2C}{d\tau^2} = \left\langle x(t)x''(t+\tau)\right\rangle_t \\ &\text{(or replacing } t-\tau=\eta\ (\tfrac{\partial(t-\tau)}{\partial\tau}=-1)) \\ &= \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} \left(-x'(\eta)x'(\eta+\tau)\right)\,d\eta \\ &= -\left\langle x'(t)x'(t+\tau)\right\rangle_t \end{aligned}\right)$$

Cross-correlation between $x(t)$ and $x''(t)$.

and

Negative of auto-correlation of $x'(t)$.

2.4 Relation between Power spectrum and Auto-correlation

Domain of signal: $(x(t) \in \mathbb{R})$

$$x(t) : \begin{cases} \neq 0 & -\frac{T}{2} \leq t \leq \frac{T}{2} \\ = 0 & \text{elsewhere} \end{cases}$$

Fourier transform (integral):

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, d\omega \left(= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) e^{-j\omega t}\, d\omega \right)$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$$

Auto-correlation function:

$$\begin{aligned}
C(\tau) &= \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)x(t+\tau)\, dt \\
&= \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \left[\frac{1}{2} \int_{-\infty}^{\infty} X^*(\omega) e^{-j\omega t}\, d\omega \right] \left[\frac{1}{2} \int_{-\infty}^{\infty} X(\omega') e^{j\omega'(t+\tau)}\, d\omega' \right] dt \\
&= \lim_{T\to\infty} \frac{1}{2\pi T} \int_{-\infty}^{\infty} X^*(\omega) e^{j\omega\tau} \int_{-\infty}^{\infty} X(\omega') \underbrace{\left(\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j(\omega'-\omega)t}\, dt \right)}_{\delta(\omega'-\omega)} d\omega'\, d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{\lim_{T\to\infty} \frac{X^*(\omega)X(\omega)}{T}}_{=S(\omega)} e^{j\omega\tau}\, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) e^{j\omega\tau}\, d\omega
\end{aligned}$$

$C(\tau)$ is identical to the inverse Fourier transform of $S(\omega)$.

Wiener-Khintchine's theorem

$$C(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)e^{j\omega\tau}\,d\omega \quad (10)$$

$$S(\omega) = \int_{-\infty}^{\infty} C(\tau)e^{-j\omega\tau}\,d\tau \quad (11)$$

$$\begin{array}{ccc} x(t) & \xrightarrow{\langle x(t)x(t+\tau)\rangle_t} & C(\tau) \\ \downarrow\uparrow \text{FT} & & \downarrow\uparrow \text{FT} \\ X(\omega) & \xrightarrow{\langle X(\omega)X^*(\omega)\rangle_t} & S(\omega) \end{array}$$

$\langle \cdots \rangle_t$: Time average when $T \to \infty$.

2.5 Cross-spectrum and Cross-correlation

$$C_{xy}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xy}(\omega)e^{j\omega\tau}\,d\omega \quad (12)$$

$$S_{xy}(\omega) = \int_{-\infty}^{\infty} C_{xy}(\tau)e^{-j\omega\tau}\,d\tau \quad (13)$$

Cross-spectral density

$$S_{xy}(\omega) = \lim_{T\to\infty}\frac{X^*(\omega)Y(\omega)}{T} = \langle X^*(\omega)Y(\omega)\rangle_t \quad (14)$$

$$\left(\begin{aligned}
C_{xy}(\tau) &= \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)y(t+\tau)\,dt = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x^*(t)y(t+\tau)\,dt \qquad (\because x(t), y(t) \in \mathbb{R})\\
&= \lim_{T\to\infty}\frac{1}{T}\int_{-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{\omega} X^*(\omega)e^{-j\omega t}\,d\omega\right)\left(\frac{1}{2\pi}\int_{\omega'} Y(\omega)e^{j\omega'(t+\tau)}\,d\omega'\right)dt\\
&= \lim_{T\to\infty}\frac{1}{2\pi T}\int_{\omega} X^*(\omega)e^{j\omega\tau}\int_{\omega'} Y(\omega')\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j(\omega'-\omega)t}\,dt\right)d\omega'\,d\omega\\
&= \lim_{T\to\infty}\frac{1}{2\pi T}\int_{\omega} X^*(\omega)e^{j\omega\tau}\int_{\omega'} Y(\omega')\delta(\omega'-\omega)\,d\omega'\,d\omega = \lim_{T\to\infty}\frac{1}{2\pi T}\int_{\omega} X^*(\omega)Y(\omega)e^{j\omega\tau}\,d\omega\\
&= \frac{1}{2\pi}\int_{\omega}\lim_{T\to\infty}\frac{X^*(\omega)Y(\omega)}{T}e^{j\omega\tau}\,d\omega = \frac{1}{2\pi}\int_{\omega} S_{xy}(\omega)e^{j\omega\tau}\,d\omega
\end{aligned}\right)$$

Characteristics of Cross-spectrum

$$C_{xy}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)y(t+\tau)\,dt = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} y(t)x(t-\tau)\,dt = C_{yx}(-\tau)$$

$$\to C_{xy}(\tau) \in \mathbb{R} \qquad (\because x(t), y(t) \in \mathbb{R})$$

$$S_{xy}(\omega) = \mathcal{F}\left\{C_{xy}(\tau)\right\}$$

$$C_{xy}(-\tau) = C_{yx}(\tau) \tag{15}$$

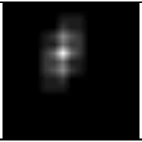
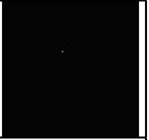
$$S_{xy}(-\omega) = S^*_{xy}(\omega) \tag{16}$$

$$S_{yx}(\omega) = S^*_{xy}(\omega) \tag{17}$$

$$S_{xy}(-\omega) = S_{yx}(\omega) \tag{18}$$

Evaluation of translation

$g(\boldsymbol{r}) = f(\boldsymbol{r} + \boldsymbol{\Delta}), \quad \boldsymbol{\Delta} = (-8, 16)$ (Image size:128x128)

$f(\boldsymbol{r})$	$g(\boldsymbol{r})$	$C_{fg}(\boldsymbol{r})$ (w/o POC)	$\widehat{C}_{fg}(\boldsymbol{r})$ (w/ POC)
			
		$(-8, 16)$	$(-8.0, 16.0)$

$$\left(\begin{aligned} C_{fg} &= \mathcal{F}^{-1}\{\langle F^* G\rangle_t\} \\ \widehat{C}_{fg} &= \mathcal{F}^{-1}\left\{\left\langle \widehat{F}^* \widehat{G}\right\rangle_t\right\} \\ \widehat{F} &= \frac{F}{|F|}, \quad \widehat{G} = \frac{G}{|G|} \\ \widehat{F}^* \widehat{G} &= e^{i(\phi_G - \phi_F)} \end{aligned}\right)$$

POC:Phase Only cross-Correlation

In the case of $g(\boldsymbol{r}) = f(\boldsymbol{r} + \boldsymbol{\Delta})$

$$G(\boldsymbol{k}) = e^{i\boldsymbol{k}\cdot\boldsymbol{\Delta}} F(\boldsymbol{k})$$

$$\widehat{F}^*(\boldsymbol{k})\widehat{G}(\boldsymbol{k}) = e^{i\boldsymbol{k}\cdot\boldsymbol{\Delta}}$$

($\phi_F(\boldsymbol{k})$ is canceled.)

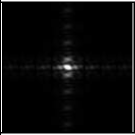
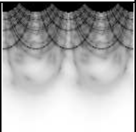
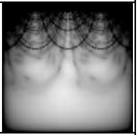
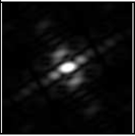
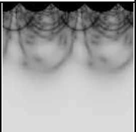
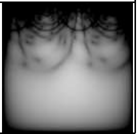
$$\begin{aligned} \widehat{C}_{fg}(\boldsymbol{r}) &= \mathcal{F}^{-1}\left\{\left\langle e^{i\boldsymbol{k}\cdot\boldsymbol{\Delta}}\right\rangle_r\right\} \\ &= \lim_{L\to\infty} \frac{1}{L^2}\frac{1}{(2\pi)^2}\int e^{i\boldsymbol{k}\cdot(\boldsymbol{\Delta}-\boldsymbol{r})}\, d\boldsymbol{k}^2 \\ &= \lim_{L\to\infty} \frac{1}{L^2}\delta(\boldsymbol{r} - \boldsymbol{\Delta}) \rightarrow \text{Sharp peak} \end{aligned}$$

Evaluation of rotation and scaling

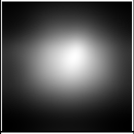
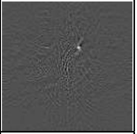
$$g(\boldsymbol{r}) = f(\alpha\boldsymbol{\Theta}\cdot\boldsymbol{r} + \boldsymbol{\Delta}), \quad \boldsymbol{\Delta} = (-8, 16),$$

$$\alpha = \frac{1}{2}, \; \boldsymbol{\Theta} = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix}, \; \theta_1 = 30\,\mathrm{deg}$$

(Image size:128x128)

(x, y)	Cartesian (k_x, k_y)	log-polar $(k_\theta, \log\lvert\boldsymbol{k}\rvert)$	Hanning-Win. $(k_\theta, \log\lvert\boldsymbol{k}\rvert)$
$f(\boldsymbol{r})$	$I_F(\boldsymbol{k}) = \lvert F(\boldsymbol{k})\rvert$		$\mathcal{H}\{I_F(\boldsymbol{k})\}$
			
$g(\boldsymbol{r})$	$I_G(\boldsymbol{k}) = \lvert G(\boldsymbol{k})\rvert$		$\mathcal{H}\{I_G(\boldsymbol{k})\}$
			
		(log scale $[0.05, 500]$)	(log scale $[0.05, 500]$)

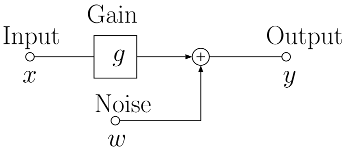
$$C_{I_F I_G} = \mathcal{F}^{-1}\left\{(\mathcal{F}\{\mathcal{H}\{I_F\}\})^* \mathcal{F}\{\mathcal{H}\{I_G\}\}\right\}$$

	Cross-corr. of I_F and I_G $(k_\theta, \log\lvert\boldsymbol{k}\rvert)$ $C_{I_F I_G}(\boldsymbol{k})$	
	w/o POC	w/ POC
		
k_θ [deg]	19.6	29.4
$\log\lvert\boldsymbol{k}\rvert$	-0.343	-0.682
$\frac{1}{\lvert\boldsymbol{k}\rvert}$ [times]	0.710	0.505

Observation Model of Additive Noise

3. Noise
3.1 Observation Model of Additive Noise

Model of Additive Noise



$$y = gx + w$$

N times measurements

$$(y^{(n)},\, n \in \{1, \cdots, N\})$$

$$y^{(n)} = gx^{(n)} + w^{(n)}$$

$$\left(\begin{array}{ll} y^{(n)} & : \text{ known} \\ x^{(n)}, w^{(n)} & : \text{ unknown} \end{array} \right)$$

- x and w are independent
- Definition of Expected value and Variance

  Expected Value :

  $$\mathrm{E}[f] \equiv \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f^{(n)} \equiv \overline{f}$$

  Variance :

  $$\sigma_f^2 \equiv \mathrm{E}\left[\left(f - \overline{f} \right)^2 \right]$$

- What is the relation between $\overline{x}$, $\overline{w}$, and $\overline{y}$, or that between σ_x^2, σ_w^2, and σ_y^2?

Statistics of Observed value in the additive noise model

$y = gx + w$

- Ave. (Exp. Val.)

$$\overline{y} = \mathrm{E}\left[y\right] = \mathrm{E}\left[gx + w\right] = \lim_{N\to\infty} \frac{1}{N}\left(\sum gx^{(n)} + w^{(n)}\right) = g\mathrm{E}\left[x\right] + \mathrm{E}\left[w\right] = g\overline{x} + \overline{w}$$

- Var.

$$\sigma_y^2 = \mathrm{E}\left[\left((gx+w) - \overline{y}\right)^2\right] = \mathrm{E}\left[\left((gx+w) - (g\overline{x} + \overline{w})\right)^2\right] = \mathrm{E}\left[\left(g(x-\overline{x}) + (w-\overline{w})\right)^2\right]$$

$$= g^2 \underbrace{\mathrm{E}\left[(x-\overline{x})^2\right]}_{=\sigma_x^2} + 2\underbrace{\mathrm{E}\left[(x-\overline{x})(w-\overline{w})\right]}_{=0\ (\mathrm{E}[xw]=\overline{x}\cdot\overline{w}\text{より})} + \underbrace{\mathrm{E}\left[(w-\overline{w})^2\right]}_{=\sigma_w^2} = g^2\sigma_x^2 + \sigma_w^2$$

In general, $\overline{w} = 0$, $\sigma_w^2 > 0$. When $N \to \infty$,

- $\overline{y} = g\overline{x}$

  $\rightarrow$ Effect of noise can be removed.
- $\sigma_y^2 = g^2\sigma_x^2 + \sigma_w^2 > g^2\sigma_x^2$

  $\rightarrow$ The variance caused by noise cannot be removed.

3.2 Classification of random signal

- Stationary
  - Ergodic
  - Non-Ergodic
- Not stationary
  - Non-ergodic

- Multiple measurements of time varying signal $(f^{(n)}(t),\ n \in \{1, \cdots, N\})$

$$\overline{f}(t) = E[f(t)] = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f^{(n)}(t)$$

$$C(t,\tau) = E[f(t)f(t+\tau)]$$

$$= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f^{(n)}(t)f^{(n)}(t+\tau)$$

  - Stationary :
    $\overline{f}(t) = \overline{f'}$, $C(t,\tau) = C'(\tau)$
    (Independent of t)

- One of time varying signal

$$\langle f \rangle^{(n)} = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f^{(n)}(t)\, dt$$

$$C^{(n)}(\tau)$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f^{(n)}(t) f^{(n)}(t+\tau)\, dt$$

  - Ergodic :
    $\langle f \rangle^{(n)} = \overline{f'}$, $C^{(n)}(\tau) = C'(\tau)$

3.3 Spectrum of Noise
White Noise

Stationary Ergodic random signal

$$(\langle n(t)\rangle = 0, \sigma_n^2 = \left\langle n^2(t)\right\rangle = \overline{n^2} > 0)$$

White Noise : Independent at $\tau \neq 0$.

- Auto correlation : $C(\tau) = \langle n(t)n(t+\tau)\rangle = \lim_{\Delta t \to 0} \overline{n^2}\delta(\tau)\Delta t$ (1)

(Δt is added to match the dimension.)

- Power Spectrum : $S(\omega) = \int \overline{n^2}\delta(\tau)\Delta t e^{-j\omega\tau}\, d\tau = \overline{n^2}\Delta t$ (2)

$\rightarrow$ Spectrum is constant.⇔White

- Cross correlations between other signals are 0.

$$C_{nf}(\tau) = \langle n(t)f(t+\tau)\rangle = \underbrace{\langle n(t)\rangle}_{=0}\langle f(t+\tau)\rangle = 0 \qquad (3)$$

Brownian Noise($1/f^2$ Noise)

- Markov Process(Affected by the previous point(neighbors))

$$r(t+\Delta t) = \rho r(t) + n(t) \quad (4)$$

$\left(0 < \rho < 1,\ \Delta t > 0,\ n(t)\right.$: white noise $\left.\right)$

- Auto correlation

$$C(\tau) = \langle r(t) r(t+\tau) \rangle$$

$$\begin{aligned} C(\tau + \Delta t) &= \langle r(t) r(t+\tau+\Delta t) \rangle \\ &= \langle r(t) \{ \rho r(t+\tau) + n(t+\tau) \} \rangle \\ &= \rho \langle r(t) r(t+\tau) \rangle + \langle r(t) n(t+\tau) \rangle \\ &= \rho C(\tau) \end{aligned}$$

Compare Taylor series expansion.$(\tau > 0)$

$$C(\tau + \Delta t) = C(\tau) + \frac{dC}{d\tau}\Delta t + O(\Delta t^2)$$

$$\rightarrow \quad \rho C = C + \frac{dC}{d\tau}\Delta t$$

$$\rightarrow \quad \frac{1}{C} dC = -\underbrace{\frac{1-\rho}{\Delta t}}_{=\alpha} d\tau = -\alpha\, d\tau$$

$$C(\tau) = C_0 e^{-\alpha\tau}$$

Since $C(\tau)$ has an even property,

$$C(\tau) = C_0 e^{-\alpha|\tau|} \quad (5)$$

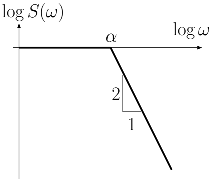
C_0 is obtained from Eq.(4).

$$\left\langle r^2(t+\Delta t) \right\rangle = \left\langle (\rho r(t) + n(t))^2 \right\rangle$$

$$C_0 = \rho^2 C_0 + \sigma_n^2$$

$$\therefore \quad C_0 = \frac{\sigma_n^2}{1-\rho^2} \quad (6)$$

- Power spectrum



$$S(\omega) = \int_{-\infty}^{0} C_0 e^{(\alpha - j\omega)t}\,dt + \int_{0}^{\infty} C_0 e^{(-\alpha - j\omega)t}\,dt$$
$$= C_0\left(\frac{1}{\alpha - j\omega} + \frac{-1}{-\alpha - j\omega}\right) = \frac{2C_0\alpha}{\omega^2+\alpha^2} \qquad (7)$$

Lorentz distribution $\propto \dfrac{1}{\omega^2+\alpha^2}$

$$S(\omega) \propto \begin{cases} \text{const} & (\omega \ll \alpha) \\ \frac{1}{\omega^2} \propto \frac{1}{f^2} & (\omega \gg \alpha) \end{cases} \qquad (8)$$

- Naming
  - Red Noise ←Higher freq. (short wavelength) comp. is small.
  - Brownian noise (Not color 'brown')
    ←Spectrum of particle position with Brownian motion (Random walk)
  - Lorentzian Noise
- e.g. Thermal noise

$1/f$ Noise

Noise with $S(\omega) \propto \dfrac{1}{\omega}$ $\qquad$ (※ $|X(\omega)| \propto \dfrac{1}{\sqrt{\omega}} \neq \dfrac{1}{\omega}$)

Signals with $1/f$ noise

- Electric resistance of metal (fluctuation of num. of carriers)
- Sound from small stream of water
- pitch of grain of wood

The mechanism is not known clearly.

(Another name) Pink noise (Intermediate White and Red.)

3.4 Auto correlation of observed value.

$s'(t) = s(t) + n(t)$

$s(t)$	: True(Unknown.)	$C_s(\tau)$ is also unknown.
$n(t)$	: White Noise(Unknown.)	$C_n(\tau) = \overline{n^2}\delta(\tau)$ is known.
$s'(t)$	: Observed(Known.)	$C_{s'}(\tau)$ is also known.

$$
\begin{aligned}
C_{s'}(\tau) &= \left\langle s'(t)s'(t+\tau)\right\rangle = \langle\{s(t)+n(t)\}\{s(t+\tau)+n(t+\tau)\}\rangle \\
&= \underbrace{\langle s(t)s(t+\tau)\rangle}_{=C_s(\tau)} + \underbrace{\langle n(t)s(t+\tau)\rangle}_{=0} + \underbrace{\langle s(t)n(t+\tau)\rangle}_{=0} + \underbrace{\langle n(t)n(t+\tau)\rangle}_{=C_n(\tau)} \\
&= C_s(\tau) + \overline{n^2}\delta(\tau)\Delta t
\end{aligned}
$$

$$
\therefore C_s(\tau) = C'_s(\tau) - \overline{n^2}\delta(\tau)\Delta t
$$

If we know property of noise, we can obtain auto correlation of true.

3.5 Intensity distribution of Noise

- The terminology White or Red expresses spectrum in freq. domain. This represents periodicity of signal, it does not represent the intensity distribution of noise.
- The quantity to represent the intensity is probability distribution p. (Form of the function and Parameters (eg. standard deviation).
- Well used probability distribution :
  - Normal distribution (Gaussian distribution)
  - Uniform distribution
- To express the property of noise, both the spectrum and distribution function are required.

  (eg. White Normal distributed noise (with standard deviation))

3.6 Generation of Random Noise

Ideal Random Number: No periodicity $\to$ White Noise

- Computer simulation: Random numbers are required for generation of simulated noise. Normal rand. num. are required in many cases.
- A function in most computers is only a series of Uniform Distrib.
  Uniform Random number: $X \sim U[\min, \max]$
  Implemented one in computer system is $X \sim U[0, 1]$.
  ($X \in [0, 1]$ is generated and the probability to generating them is same.)
- Methods to generate Normal Rand. from Uniform Rand.
  - Sum of plural Uniform Random numbers.
  - Coordinate transform of Uniform Random numbers
  - Use of Multi-dimensional probability distribution function

Probability of independent events

- Random Variable: X
  $X \in \{x_1, x_2, \cdots, x_n\}$ (Discrete)
  $X \in [x_{\min}, x_{\max}]$ (Continuous)
- Probability Density Function: $p(x)$
  Probability of $X = x$:
  (In cont. system, $p(x)\,dx$.)
- Cumulative Distribution Function:
  $$P(X < x) = \int_{-\infty}^{x} p(x)\,dx$$
  $$(P(X < \infty) = 1)$$
- Average (expected value) of X:
  $$\overline{x} = \int_{-\infty}^{\infty} x p(x)\,dx$$
- Average of function with X:
  $$\overline{f(X)} = \mathrm{E}\,[f(X)] = \int_{-\infty}^{\infty} f(x)p(x)\,dx$$
- Variance of X:
  $$\sigma_x^2 = \overline{(x-\overline{x})^2} = \overline{x^2} - \overline{x}^2$$
  $$\left(\begin{aligned} \overline{(x-\overline{x})^2} &= \int (x-\overline{x})^2\, p(x)\,dx \\ &= \int (x^2 - 2x\overline{x} + \overline{x}^2)\, p(x)\,dx \\ &= \underbrace{\int x^2\, p(x)\,dx}_{=\overline{x^2}} - 2\overline{x} \underbrace{\int x\, p(x)\,dx}_{=\overline{x}} + \overline{x}^2 \underbrace{\int p(x)\,dx}_{=1} \\ &= \overline{x^2} - \overline{x}^2 \end{aligned} \right)$$
- Incident Prob. Dens. Func. for two independent events:
  $$p(x_1, x_2) = p_1(x_1)\, p_2(x_2)$$
  - $\overline{x_1 + x_2} = \overline{x_1} + \overline{x_2}$
  - $\sigma^2_{x_1+x_2} = \sigma^2_{x_1} + \sigma^2_{x_2}$
  - $\overline{x_1 x_2} = \overline{x_1} \cdot \overline{x_2}$

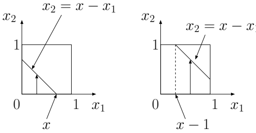
Generation of Normal random distribution using sum of uniform random numbers

Central Limit Theorem

Sum of Random Variables which obey an independent distribution converges to a Normal distribution. (There are some exceptions.)

- $X_u \sim U[0,1] \quad \rightarrow \quad p(x) = 1$
- Sum of two random number:

$$F(x) = P\left(X_1 + X_2 < x\right) = \iint_{x_1+x_2<x} dx_1\, dx_2$$



Case of $0 < x \le 1$:

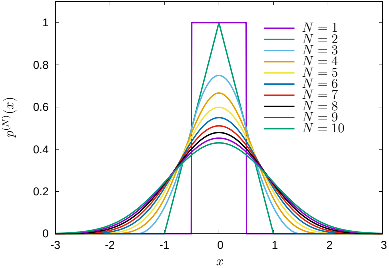
$$F(x) = \int_{x_1=0}^{x} \int_{x_2=0}^{x-x_1} dx_2\, dx_1 = \frac{x^2}{2}$$

Case of $1 < x \le 2$:

$$F(x) = \left(\int_{x_1=0}^{x-1} \int_{x_2=0}^{1} + \int_{x_1=x-1}^{1} \int_{x_2=0}^{x-x_1} \right) dx_2\, dx_1$$

$$= -\frac{x^2}{2} + 2x - 1$$

$$p(x) = \begin{cases} x & (0 < x \le 1) \\ 2 - x & (1 < x \le 2) \end{cases}$$



Generation of Random Noise

- $X_u \sim U[0,1] \quad \rightarrow \quad p(x) = 1$
- Average and Variance

$$\begin{cases} \overline{x_u} = \int_0^1 x_u \, dx_u = \frac{1}{2} \\ \sigma_{x_u}^2 = \int_0^1 (x_u - \overline{x_u})^2 \, dx_u = \overline{x_u^2} - \overline{x_u}^2 \\ \qquad = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \end{cases}$$

- $X = \sum_{n=1}^{N} X_u$

$$\begin{cases} \overline{x} = \sum_{n=1}^{N} \overline{x_u} = \frac{N}{2} \\ \sigma_x^2 = \sum_{n=1}^{N} \sigma_{x_u}^2 = \frac{N}{12} \end{cases}$$

When $N = 12$,

$$X = \sum_{n=1}^{12} X_u - 6 \sim N[0,1]$$

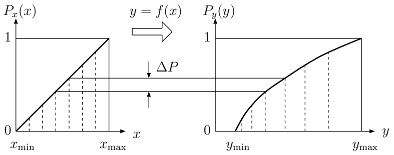
is obtained as a Normal distribution.

($N[\bar{x}, \sigma_x^2]$:
Normal distribution
with average, $\bar{x}$
with variance, σ_x^2)

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{(x-\bar{x})^2}{2\sigma_x^2}}$$

Arbitrary random distribution using coordinate transform

Coordinate change using $y = f(x)$ from $P_x(X < x)$ to $P_y(Y < y)$.



$\Delta P = p_x(x)\,dx$ $\qquad$ $\Delta P = p_y(y)\,dy$

$$p_x(x)\,dx = p_y(y)\,dy$$

We can obtain the transformation $f(x)$ from $X \sim U[0,1]$ ($p_x(x) = 1$) so that $p_y(y)$ satisfies above the Eq.

- Exponential distribution

$$p_y(y) = e^{-y}$$
$$dx = e^{-y}\,dy$$
$$\to x = -e^{-y} \to y = -\log x$$

> The exponential random distribution can obtained by transform with $y = -\log x$ where x is uniform random number.

- Randoms obeying Normal dist.

$$p_y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$$
$$x = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{y^2}{2}}\,dy = \frac{1 + \mathrm{erf}(\frac{y}{2})}{2}$$
$$\to y = 2\,\mathrm{erf}^{-1}(2x - 1)$$

erf^{-1} is not implemented in computers.

Normal random number using Box-Muller method

Multi-dimensional probability distribution:

$$p_y(y_1, y_2, \cdots)\, dy_1\, dy_2 \cdots = p_x(x_1, x_2, \cdots)\, dx_1\, dx_2 \cdots$$
$$= p_x(x_1, x_2, \cdots)|J|\, dy_1\, dy_2 \cdots$$

Assume x_1, x_2 are independent.

$X_1, X_2 \sim U[0,1]$

$(p_x(x_1, x_2) = p_x(x_1)p_x(x_2) = 1)$

$$\begin{cases} y_1 = \sqrt{-2\log x_1}\cos(2\pi x_2) \\ y_2 = \sqrt{-2\log x_1}\sin(2\pi x_2) \end{cases} \quad (1)$$

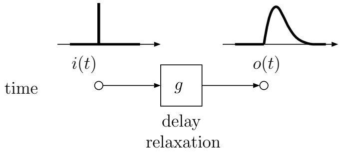
x_1 and x_2 are

$$\begin{cases} x_1 = e^{-\frac{y_1^2+y_2^2}{2}} \\ x_2 = \frac{1}{2\pi}\tan^{-1}\left(\frac{y_2}{y_1}\right) \end{cases}$$

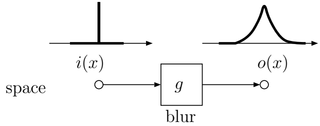
$$|J| = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \frac{1}{2\pi}e^{-\frac{y_1^2+y_2^2}{2}}$$
$$= \left(\frac{1}{\sqrt{2\pi}}e^{-\frac{y_1^2}{2}}\right)\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{y_2^2}{2}}\right)$$
$$= p_y(y_1)p_y(y_2)$$

The pair (y_1, y_2) has Normal distribution with $N[0,1]$, which is obtained by Eq. (1) using a pair of (x_1, x_2) with $U[0,1]$.

4. Convolution and Response Function



$$o(t) = \int_0^{\infty} g(\tau) i(t-\tau)\, d\tau = \int_{-\infty}^{t} i(\tau) g(t-\tau)\, d\tau \equiv g(t) * i(t) \equiv (g*i)(t)$$



$$o(x) = \int_{-\infty}^{\infty} g(\xi) i(x-\xi)\, d\xi = \int_{-\infty}^{\infty} i(\xi) g(x-\xi)\, d\xi \equiv g(x) * i(x) \equiv (g*i)(x)$$

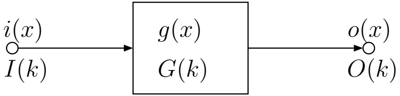
- Since input is considered as reason, and output as result, the output is considered as integral of the input.
- The response does not depend on absolute time. It only depend on the time difference between input and output.
- The output is an integral of the product between input and weight depending time difference.
- The difference of time and spatial domain is only the region of integral. (Causality)

  ※ The convolution is similar to the cross-correlation but sign of the argument is inverted.

4.1 Convolution theorem

Convolution theorem

If $o(x)$ is represented as a convolution of $g(x)$ and $i(x)$,



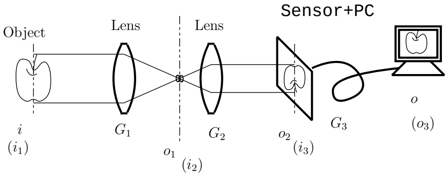
$$o(x) = g(x) * i(x)$$
$$O(k) = G(k) \cdot I(k)$$

$$\begin{cases} F(k) = \displaystyle\int f(x)\, e^{-ikx}\, dx \\ f(x) = \dfrac{1}{2\pi} \displaystyle\int F(k)\, e^{+ikx}\, dk \end{cases}$$

$$o(x) = \int_\xi g(\xi)\, i(x-\xi)\, d\xi$$

$$\begin{aligned} O(k) &= \int_x o(x)\, e^{-ikx}\, dx \\ &= \int_x \left(\int_\xi g(\xi)\, i(x-\xi)\, d\xi \right) e^{-ikx}\, dx \\ &= \int_\xi g(\xi) \left(\int_x i(x-\xi)\, e^{-ikx}\, dx \right) d\xi \\ &= \int_\xi g(\xi) \left(\int_x i(x-\xi)\, e^{-ik(x-\xi)}\, dx\ e^{-ik\xi} \right) d\xi \\ &\qquad (x' = x - \xi) \\ &= \left(\int_\xi g(\xi)\, e^{-ik\xi}\, d\xi \right) \cdot \left(\int_{x'} i(x')\, e^{-ikx'}\, dx' \right) \\ &= G(k) \cdot I(k) \end{aligned}$$

4.2 Response Function



$$o_n(\boldsymbol{r}) = \int g_n(\boldsymbol{\xi})\, i_n(\boldsymbol{r} - \boldsymbol{\xi})\, d\boldsymbol{\xi}$$
$$= (g_n * i_n)(\boldsymbol{r}) = i_{n+1}(\boldsymbol{r})$$
$$O_n(\boldsymbol{k}) = G_n(\boldsymbol{k})\, I_n(\boldsymbol{k}) = I_{n+1}(\boldsymbol{k})$$

$$o(\boldsymbol{r}) = (g_N * (\cdots * (g_1 * i)))(\boldsymbol{r}), \qquad O(\boldsymbol{k}) = \left(\prod_{n=1}^{N} G_N(\boldsymbol{k})\right) I(\boldsymbol{k})$$

- The response function of a whole system equals to the products of response functions of each components.
- When the response function in each system is known, $i(\boldsymbol{r})$ can be obtained from $o(\boldsymbol{r})$.

Measurement of Response function

- Response func. of pin-hole $(i(\boldsymbol{r}) = \delta(\boldsymbol{r}))$

$$o(\boldsymbol{r}) = \int g(\boldsymbol{r} - \boldsymbol{\xi})\,\delta(\boldsymbol{\xi})\,d\boldsymbol{\xi} = g(\boldsymbol{r})$$
$$O(\boldsymbol{k}) = G(\boldsymbol{k})$$

  $o(\boldsymbol{r})$: Point Spread Function
  $O(\boldsymbol{k})$: Point Response Function

- Response func. of slit $(i(\boldsymbol{r}) = \delta(x))$
  $o_x(\boldsymbol{r})$: Line Spread Function
  $O_x(\boldsymbol{k})$: Line Response Function

Is there an ideal pin hole or slit?

- Response of edge $(i_e(\boldsymbol{r}) = \theta(x))$

$$\left(\theta : \text{step func.},\ \frac{d\theta}{dx} = \delta(x)\right)$$

$$o_e(x) = \int g_x(x-\xi)\,\theta(\xi)\,d\xi = \int g_x(\xi')\,\theta(x-\xi')\,d\xi'$$

$$\begin{aligned}
\frac{do_e(x)}{dx} &= \frac{d}{dx}\int g_x(\xi')\,\theta(x-\xi')\,d\xi' \\
&= \int g_x(\xi')\underbrace{\frac{d\theta(x-\xi')}{d(x-\xi')}}_{\delta(x-\xi')}\underbrace{\frac{d(x-\xi')}{dx}}_{=1}\,d\xi' \\
&= \int g_x(\xi')\,\delta(x-\xi')\,d\xi' = g_x(x)
\end{aligned}$$

$$\begin{aligned}
G_x(k_x) &= \int g_x(x)\,e^{-ik_x x}\,dx = \int \frac{do_e}{dx}\,e^{-ik_x x}\,dx \\
&= \underbrace{\Big[o_e(x)\,e^{-ik_x x}\Big]_{-\infty}^{\infty}}_{=0\quad (\because o_e(\pm\infty)=0)} + ik_x \int o_e(x)\,e^{-ik_x x}\,dx \\
&= ik_x O_e(k_x)
\end{aligned}$$

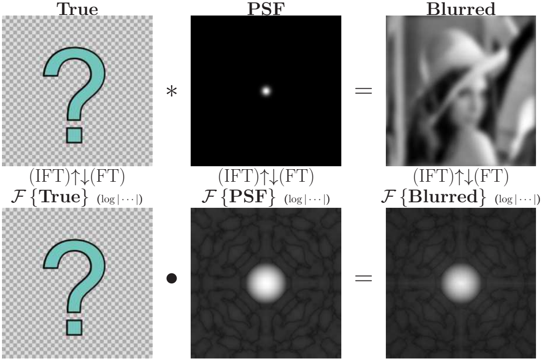
$O_e(\boldsymbol{k})$: Edge Response Function

4.3 Deconvolution

A blurred image and the PSF of whole system is known.

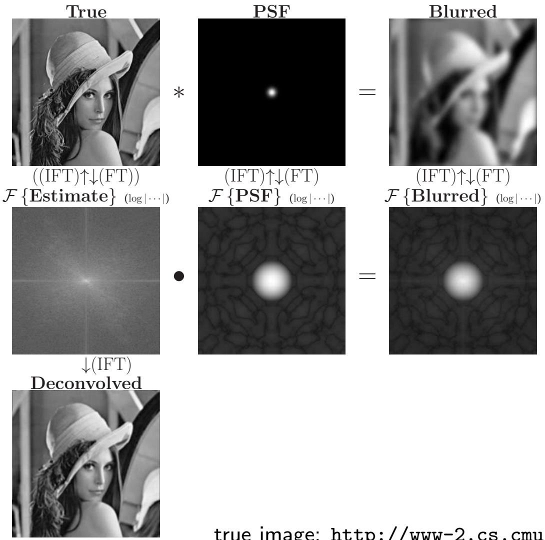
$\rightarrow \mathcal{F}\{\text{Blurred}\}$ and $\mathcal{F}\{\text{PSF}\}$ are also known.

We wish to reconstruct the true image.



$$\text{True'} = \mathcal{F}^{-1}\left\{\frac{\mathcal{F}\{\text{Blurred}\}}{\mathcal{F}\{\text{PSF}\}}\right\} \quad \text{(Deconvolution)}$$

Example of reconstruction from blurred image



In $\mathcal{F}\{\text{Blurred}\}/\mathcal{F}\{\text{PSF}\}$, it has a divergence if the denominator is small.

↓

Since special treatments are applied (discuss in later days) to avoid divergence, True $\neq$ Deconvoluted

(※ In this example, it reconstructed only from the Blurred image, i.e., PSF is also unknown.)

true image: `http://www-2.cs.cmu.edu/~chuck/lennapg/lena_std.tif`

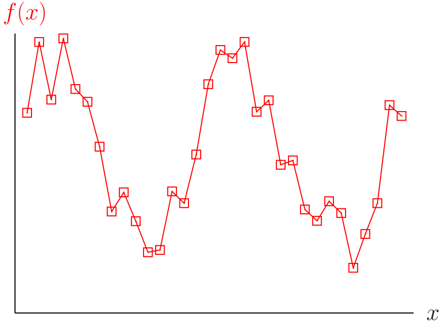
5. Noise Reduction using Moving average

Observed data including noise

$f(x_i) = \tilde{f}(x_i) + n(x_i)$

$(i \in \{1, \cdots, N_{all}\})$

Ave. Method	Num. of Ave.
whole data	1
Ave. for M groups	$M < N_{all}$
Moving Average	$N'_{all} \sim N_{all}$



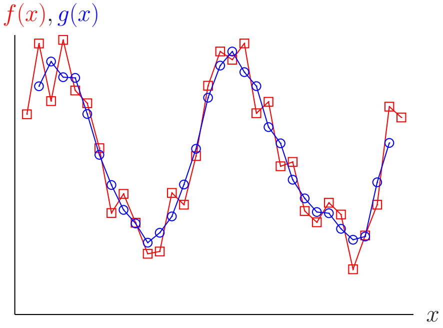
5. Noise Reduction using Moving average

Observed data including noise

$f(x_i) = \tilde{f}(x_i) + n(x_i)$

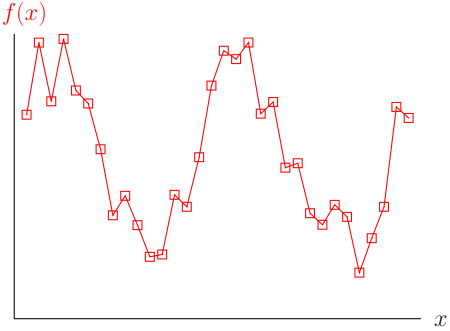
$(i \in \{1, \cdots, N_{all}\})$

Ave. Method	Num. of Ave.
whole data	1
Ave. for M groups	$M < N_{all}$
Moving Average	$N'_{all} \sim N_{all}$



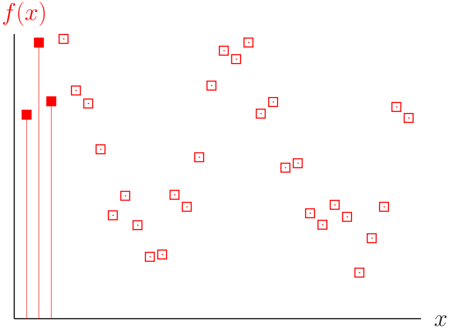
5.1 Moving Average

Averages are computed while replacing a part of samples.



5.1 Moving Average

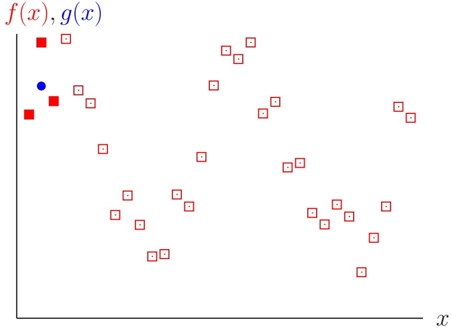
Averages are computed while replacing a part of samples.



Moving average with 3 points

5.1 Moving Average

Averages are computed while replacing a part of samples.

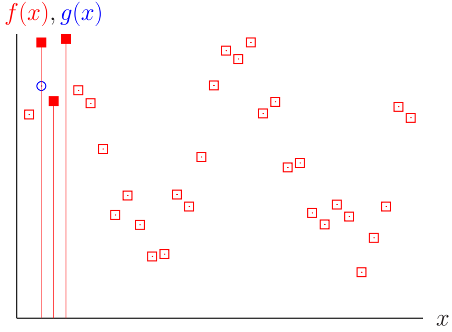


Moving average with 3 points

$$g_2 = (f_1 + f_2 + f_3)/3$$

5.1 Moving Average

Averages are computed while replacing a part of samples.

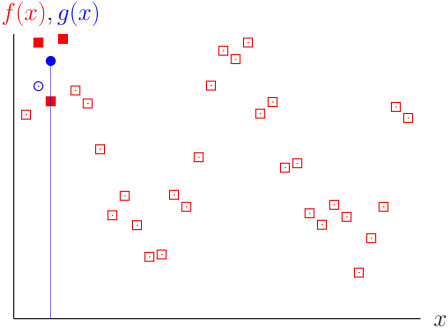


Moving average with 3 points

$$g_2 = (f_1 + f_2 + f_3)/3$$

5.1 Moving Average

Averages are computed while replacing a part of samples.

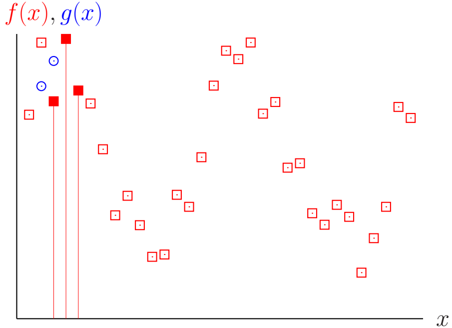


Moving average with 3 points

$$g_2 = (f_1 + f_2 + f_3)/3$$
$$g_3 = (f_2 + f_3 + f_4)/3$$

5.1 Moving Average

Averages are computed while replacing a part of samples.

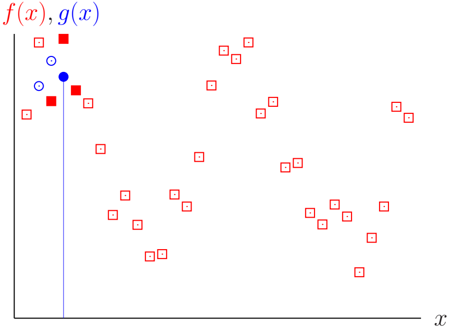


Moving average with 3 points

$$g_2 = (f_1 + f_2 + f_3)/3$$
$$g_3 = (f_2 + f_3 + f_4)/3$$

5.1 Moving Average

Averages are computed while replacing a part of samples.

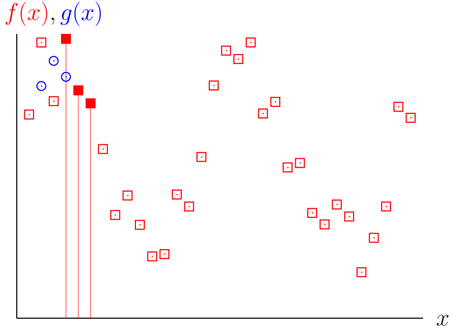


Moving average with 3 points

$$g_2 = (f_1 + f_2 + f_3)/3$$
$$g_3 = (f_2 + f_3 + f_4)/3$$
$$g_4 = (f_3 + f_4 + f_5)/3$$

5.1 Moving Average

Averages are computed while replacing a part of samples.

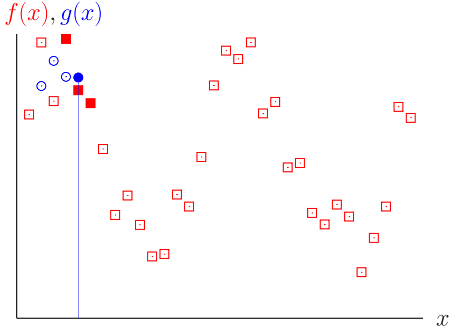


Moving average with 3 points

$$g_2 = (f_1 + f_2 + f_3)/3$$
$$g_3 = (f_2 + f_3 + f_4)/3$$
$$g_4 = (f_3 + f_4 + f_5)/3$$

5.1 Moving Average

Averages are computed while replacing a part of samples.

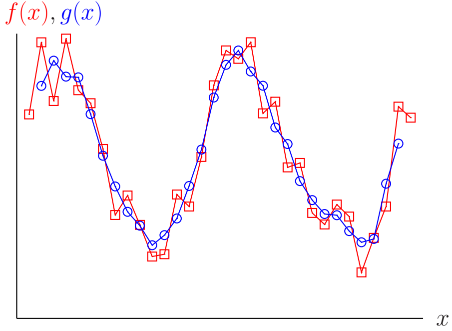


Moving average with 3 points

$$g_2 = (f_1 + f_2 + f_3)/3$$
$$g_3 = (f_2 + f_3 + f_4)/3$$
$$g_4 = (f_3 + f_4 + f_5)/3$$
$$g_5 = (f_4 + f_5 + f_6)/3$$

5.1 Moving Average

Averages are computed while replacing a part of samples.



Moving average with 3 points

$$g_2 = (f_1 + f_2 + f_3)/3$$
$$g_3 = (f_2 + f_3 + f_4)/3$$
$$g_4 = (f_3 + f_4 + f_5)/3$$
$$g_5 = (f_4 + f_5 + f_6)/3$$
$$\vdots$$
$$g_i = (f_{i-1} + f_i + f_{i+1})/3$$

$$\boxed{g_i = \sum_{m'} w_{m'} f_{i+m'} = \sum_{m} w_m f_{i-m}}$$

5.2 Relation between Moving Average and Convolution

Moving Average≡Discrete Convolution Integral

$g(x)$: averaged, $f(x)$: observed

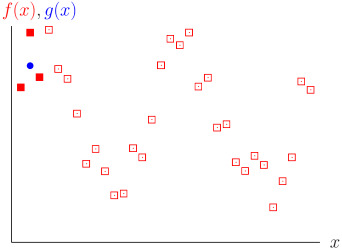
Continuous system: $$g(x) = \int_{-\infty}^{\infty} w(x') f(x - x')\, dx' \quad \text{(Convolution)}$$

Discrete system: $$g_i = \sum_{m=-\infty}^{\infty} w_m f_{i-m} \Delta x \qquad (f_i \equiv f(x_i))$$

If m is finite $(N_m = 2N + 1)$,

$$g_i = \sum_{m=-N}^{N} \widehat{w}_m f_{i-m} \quad (\widehat{w}_m = w_m \Delta x) \quad (1)$$

$$\int_{-\infty}^{\infty} w(x')\, dx' = 1 \quad \text{(Normalization)} \quad (2)$$



5.3 Simple Moving Average

Weight $w(x')$ inside the window is constant

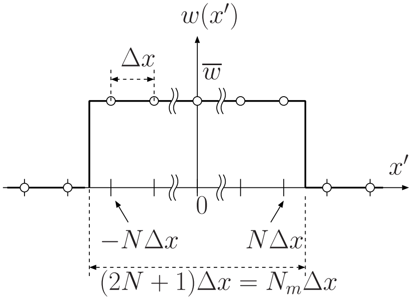
$$w(x') = \begin{cases} \text{const} = \overline{w} & \left(|x'| \leq \dfrac{N_m \Delta x}{2}\right) \\ 0 & \text{(elsewhere)} \end{cases}$$

From Eq.(2)

$$\int_{-\infty}^{\infty} w(x')\,dx' = N_m \Delta x \overline{w}$$

$$= N_m \widehat{w} = 1$$

$$\widehat{w}_m = \begin{cases} \dfrac{1}{N_m} & (|m| \leq N) \\ 0 & \text{(otherwise)} \end{cases}$$



Comparison of Noise Reduction (original signal)

Before averaging $f_i = \tilde{f}_i + n_i$

(f: obs., $\tilde{f}$: true (unique), n: noise)

(Measurement infinite times: $f_i^j \quad (j \in \{1, \cdots, \infty\})$)

- Average with respect to j:

$$\mathrm{E}[f_i] = \mathrm{E}\left[\tilde{f}_i + n_i\right] = \underbrace{\mathrm{E}\left[\tilde{f}_i\right]}_{=\tilde{f}_i} + \underbrace{\mathrm{E}[n_i]}_{=0} = \tilde{f}_i$$

- Variance:

$$\begin{aligned} \sigma_{f_i}^2 &= \mathrm{E}\left[(f_i - \mathrm{E}[f_i])^2\right] \\ &= \mathrm{E}\left[((\tilde{f}_i + n_i) - \tilde{f}_i)^2\right] \\ &= \mathrm{E}\left[n_i^2\right] = \sigma_{n_i}^2 = \sigma_n^2 \end{aligned}$$

(Another method to evaluate variance

If two signals a and b have no correlations,

$$\sigma_{a\pm b}^2 = \sigma_a^2 + \sigma_b^2.$$

$\tilde{f}_i$ and n_i have no correlations, and $\sigma_{\tilde{f}_i}^2 = 0$ because $\tilde{f}$ is unique.

$\therefore \sigma_{f_i}^2 = \sigma_{\tilde{f}_i}^2 + \sigma_{n_i}^2 = \sigma_n^2$)

Comparison of Noise Reduction (Moving Average)(1)

Moving Average $$g_i = \frac{1}{N_m}\sum_m f_{i-m} \equiv \langle f_i \rangle_m$$

$$f_i = \tilde{f}_i + n_i \rightarrow g_i = \left\langle \tilde{f}_i \right\rangle_m + \langle n_i \rangle_m$$

- Average with respect to j:

$$\mathrm{E}\left[g_i\right] = \mathrm{E}\left[\left\langle \tilde{f}_i \right\rangle_m\right] + \mathrm{E}\left[\langle n_i \rangle_m\right] = \left\langle \underbrace{\mathrm{E}\left[\tilde{f}_i\right]}_{=\tilde{f}_i} \right\rangle_m + \left\langle \underbrace{\mathrm{E}\left[n_i\right]}_{=0} \right\rangle_m = \left\langle \tilde{f}_i \right\rangle_m$$

- Variance:

$$\sigma_{g_i}^2 = \mathrm{E}\left[(g_i - \mathrm{E}\left[g_i\right])^2\right] = \mathrm{E}\left[\left\{\left(\left\langle \tilde{f}_i \right\rangle_m + \langle n_i \rangle_m\right) - \left\langle \tilde{f}_i \right\rangle_m\right\}^2\right] = \mathrm{E}\left[\langle n_i \rangle_m^2\right]$$

$$= \mathrm{E}\left[\left(\frac{\sum_m n_{i-m}}{N_m}\right) \cdot \left(\frac{\sum_m n_{i-m}}{N_m}\right)\right] = \mathrm{E}\left[\frac{\sum_m \sum_{m'} n_{i-m} n_{i-m'}}{N_m^2}\right]$$

$$= \frac{1}{N_m^2}\sum_m \sum_{m'} \mathrm{E}\left[n_{i-m} n_{i-m'}\right]$$

The variance depends on the auto-correlation of noise.

Comparison of Noise Reduction (Moving Average)(2)

Variation depending noise property

$$\sigma_{g_i}^2 = \frac{1}{N_m^2} \sum_m \sum_{m'} \mathrm{E}\left[n_{i-m} n_{i-m'}\right]$$

- Case of White noise:
  (n_{i-m} and $n_{i-m'}$ are independent.)

$$\mathrm{E}\left[n_{i-m} n_{i-m'}\right] = \sigma_n^2 \delta_{m,m'}$$

$$\sigma_{g_i}^2 = \frac{\sigma_n^2}{N_m^2} \underbrace{\sum_m \sum_{m'} \delta_{m,m'}}_{N_m} = \frac{\sigma_n^2}{N_m}$$

- Case of Low frequency noise
  ($n_{i-m} \simeq n_{i-m'}$)

$$\mathrm{E}\left[n_{i-m} n_{i-m'}\right] \simeq \sigma_n^2$$

$$\sigma_{g_i}^2 \simeq \frac{\sigma_n^2}{N_m^2} \underbrace{\sum_m \sum_{m'} 1}_{N_m^2} = \sigma_n^2$$

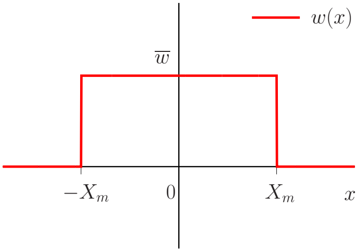
Comparison of Noise Reduction (Summary)

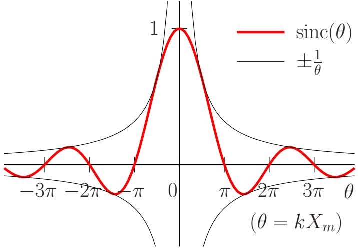
	Original	Moving Average	
Ave.	$\tilde{f}_i$	$\left\langle \tilde{f}_i \right\rangle_m$ (Smoothed True signal (drawback))	
Var.	σ_n^2	White Noise (Best)	Low frequency noise (Worst)
		$\dfrac{\sigma_n^2}{N_m}$ reduced to $1/N_m$times (advantage)	σ_n^2 Not reduced

Spectral Gain of simple moving average

Fourier Transform of $w(x')$

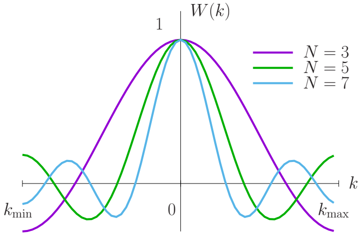
$$w(x') = \overline{w} \equiv \frac{1}{2X_m}$$
$$\left(|x'| \leq X_m = \frac{N_m \Delta x}{2}\right)$$
$$W(k) = \int_{-\infty}^{\infty} w(x') e^{-ikx'}\,dx'$$
$$= \frac{1}{2X_m} \int_{-X_m}^{X_m} e^{-ikx'}\,dx'$$
$$= \frac{1}{-2ikX_m}\left(e^{-ikX_m} - e^{+ikX_m}\right)$$
$$= \frac{\sin(kX_m)}{kX_m} = \mathrm{sinc}(kX_m)$$





Spectral Gain of simple moving average (cont.)

$$g(x) = w(x) * f(x)$$
$$G(k) = W(k) \cdot F(k)$$
$$W(k) = \mathrm{sinc}(kX_m)$$
$$(X_m = \frac{N_m \Delta x}{2})$$



Moving average can filtrate higher frequency components.

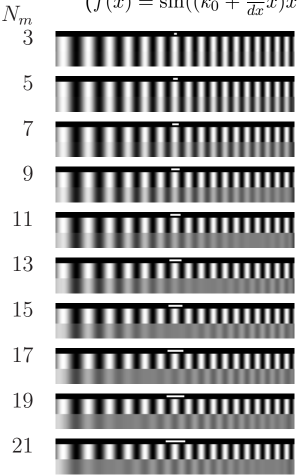
However,
$$G(k) = W(k)F(k) = W(k)\left[\widetilde{F}(k) + N(k)\right]$$
$$= W(k)\widetilde{F}(k) + W(k)N(k)$$

Since the higher frequency components of the true signal, $\tilde{f}(x)$, are reduced as well as those of the noise ($n(x)$), the true signal is distorted.

5.4 Distortion and Spurious Resolution

Distortion of true signal by moving average

$(f(x) = \sin((k_0 + \frac{dk}{dx}x)x),\ \sigma_n^2 = 0)$



- With increasing number of sampling N_m, intensity of higher frequency components become smaller.

  ↓

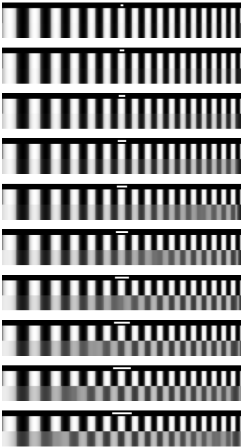
  The choice of appropriate N_m is important.
- When N_m is large, intensity becomes larger at higher frequency but those patterns are inverted. (Spurious resolution)

5.4 Distortion and Spurious Resolution

Distortion of true signal by moving average

$(f(x) = \sin((k_0 + \frac{dk}{dx}x)x),\ \sigma_n^2 = 0)$

N_m: 3, 5, 7, 9, 11, 13, 15, 17, 19, 21



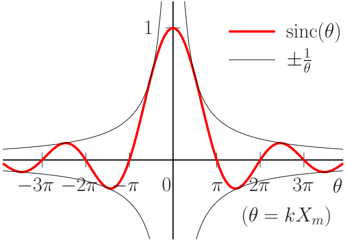
(Contrast is enhanced)

- With increasing number of sampling N_m, intensity of higher frequency components become smaller.

  ↓

  The choice of appropriate N_m is important.
- When N_m is large, intensity becomes larger at higher frequency but those patterns are inverted. (Spurious resolution)

Cause of Spurious Resolution



With increasing k, $W(k)$ is reduced.
$W(k)$ becomes 0 at $kX_m = \pi$.
After that it becomes negative.
$W(k) < 0 \rightarrow$ Inversion of intensity.

The spurious resolution is found in other filtering.

5.5 Multiple moving average

In the case of applying two times,

$$g_i = \sum_m w_m f_{i-m}$$

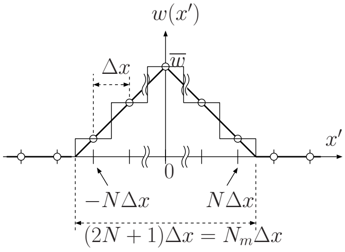
$$h_i = \sum_{m'} w_{m'} g_{i-m'}$$

$$(g_{i-m'} = \sum_m w_m f_{i-m'-m})$$

$$= \sum_{m'=-N}^{N} \sum_{m=-N}^{N} w_{m'} w_m f_{i-m'-m}$$

When $N_m = 3(N = 1), w_m = w'_m = 1/3$

$$g_i = \frac{f_{i-2} + 2f_{i-1} + 3f_{i-1} + 2f_{i+1} + f_{i+2}}{9}$$



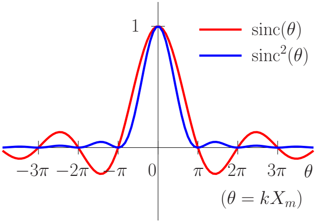
Applying multiple moving average is equivalent to moving average with which central weight is larger than neighboring points.

Spectral Gain of Multiple Moving Average

In the case of two times,

$$g(x) = \int_{-\infty}^{\infty} w(x')f(x-x')\,dx' \quad \rightarrow \quad G(k) = W(k)F(k)$$

$$h(x) = \int_{-\infty}^{\infty} w(x')g(x-x')\,dx' \quad \rightarrow \quad H(k) = W(k)G(k) = \underline{W^2(k)}F(k)$$



- Further reduction of higher frequency components is applied.
- No spurious resolution.

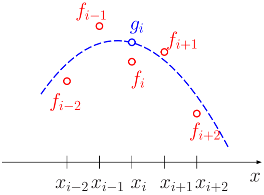
5.6 Higher order Moving average (Savitzky-Golay filter)

For each i,

- Represent the smoothed function $g_i(x_j)$ by a power series expansion.
  $(j \in \{i-N, \cdots, i+N\})$

  (In the case of second order expansion,)

  $$\begin{aligned} g_i(x_j) &= a_i(x_j - x_i)^2 + b_i(x_j - x_i) + c_i \\ &= a_i \Delta x_{j,i}^2 + b_i \Delta x_{j,i} + c_i \end{aligned}$$



- Using by the least square method, determin the parameter a_i, b_i, and c_i which are the parameter of the fitting function $g_i(x_j)$.
- The moving average at the point i corresponds to the value of the fitting function at the $x_j = x_i$; i.e $g_i(x_i) = c_i$.

Least square fitting to the parabolic function

Fitting function

$$g_j = a\Delta x_j^2 + b\Delta x_j + c \qquad (\text{(omit } i))$$

Minimize Average of square residual, E:

$$E(a,b,c) = \frac{1}{N_m}\sum_{j=i-N}^{i+N}(g_j - f_j)^2 \equiv \overline{(g_j - f_j)^2}$$

$$\text{minimize } E(a,b,c) \quad \Longleftrightarrow \quad \frac{\partial E}{\partial \xi} = 0 \quad (\xi \in \{a,b,c\})$$

$$\left(\begin{array}{c} \frac{\partial E}{\partial \xi} = \frac{\partial}{\partial \xi}\overline{(g_j(\xi) - f_j)^2} = 2\overline{(g_j(\xi) - f_j)\frac{\partial g_j(\xi)}{\partial \xi}} \\ \frac{\partial g_j}{\partial a} = \Delta x_j^2,\ \frac{\partial g_j}{\partial b} = \Delta x_j,\ \frac{\partial g_j}{\partial c} = 1 \end{array} \right)$$

$$\begin{pmatrix} \overline{\Delta x_j^4} & \overline{\Delta x_j^3} & \overline{\Delta x_j^2} \\ \overline{\Delta x_j^3} & \overline{\Delta x_j^2} & \overline{\Delta x_j} \\ \overline{\Delta x_j^2} & \overline{\Delta x_j} & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \overline{f_j \Delta x_j^2} \\ \overline{f_j \Delta x_j} \\ \overline{f_j} \end{pmatrix}$$

Least square fitting to the power series

Fitting function

$$\widetilde{f}(x;\boldsymbol{a}) = \sum_{l=0}^{N_l} a_l x^l,$$

$$\boldsymbol{a} = (a_0, a_1, \cdots, a_{N_l})$$

Sampling points (known)

$$(x_i, f_i),\ i \in \{1, \cdots, N\}$$

Minimize Average of square residual, E

$$E(\boldsymbol{a}) = \frac{1}{N_i}\sum_{i=1}^{N_i}(\widetilde{f}(x_i;\boldsymbol{a}) - f_i)^2$$

$$\equiv \overline{(\widetilde{f}(x_i;\boldsymbol{a}) - f_i)^2}$$

minimize $E(\boldsymbol{a}) \quad \Leftrightarrow \quad \frac{\partial E}{\partial a_l} = 0$

$$\left(\begin{aligned}\frac{\partial E}{\partial a_l} &= \frac{\partial}{\partial a_l}\overline{(\widetilde{f}_i - f_i)^2} = 2\overline{(\widetilde{f}_i - f_i)\frac{\partial \widetilde{f}_i}{\partial a_l}} \\ &= 2\left(\overline{\widetilde{f}_i x_i^l} - \overline{f_i x_i^l}\right) = 0 \quad \left(\because \frac{\partial \widetilde{f}_i}{\partial a_l} = x^l\right)\end{aligned}\right)$$

$$\therefore \qquad \sum_{l=0}^{N_l} \overline{x_i^{l+m}} a_l = \overline{f_i x_i^m}$$

$$\begin{pmatrix} \overline{x^0} & \overline{x^1} & \cdots & \overline{x^{N_l}} \\ \overline{x^1} & \overline{x^2} & \cdots & \overline{x^{N_{l+1}}} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{x^{N_l}} & \overline{x^{N_{l+1}}} & \cdots & \overline{x^{N_{2l}}} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{N_l} \end{pmatrix} = \begin{pmatrix} \overline{f_i x_i^0} \\ \overline{f_i x_i^1} \\ \vdots \\ \overline{f_i x_i^{N_l}} \end{pmatrix}$$

The parameter of the least square fitting to power series function (non-linear function) can be obtained by solving a set of linear equations.

Moving average by parabolic fitting

$$\begin{pmatrix} \overline{\Delta x_j^4} & \overline{\Delta x_j^3} & \overline{\Delta x_j^2} \\ \overline{\Delta x_j^3} & \overline{\Delta x_j^2} & \overline{\Delta x_j} \\ \overline{\Delta x_j^2} & \overline{\Delta x_j} & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \overline{f_j \Delta x_j^2} \\ \overline{f_j \Delta x_j} \\ \overline{f_j} \end{pmatrix}$$

In the case of $N_m = 5 (N = 2)$

$$\begin{pmatrix} \overline{\Delta x_j} = 0, \ \overline{\Delta x_j^3} = 0, \\ \overline{\Delta x_j^2}/\Delta^2 = \frac{(-2)^2+(-1)^2+0^2+(+1)^2+(+2)^2}{5} = \frac{2(1^2+2^2)}{5} = 2, \\ \overline{\Delta x_j^4}/\Delta^4 = \frac{2(1^4+2^4)}{5} = \frac{34}{5} \end{pmatrix}$$

$$c = \begin{vmatrix} \frac{34}{5}\Delta^4 & 0 & \overline{f_j \Delta x_j^2} \\ 0 & 2\Delta^2 & \overline{f_j \Delta x_j} \\ 2\Delta^2 & 0 & \overline{f_j} \end{vmatrix} \Big/ \begin{vmatrix} \frac{34}{5}\Delta^4 & 0 & 2\Delta^2 \\ 0 & 2\Delta^2 & 0 \\ 2\Delta^2 & 0 & 1 \end{vmatrix}$$

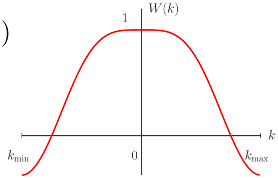
$$= \frac{\frac{68}{5}\Delta^6 \overline{f_j} - 4\Delta^4 \overline{f_j \Delta x_j^2}}{\frac{28}{5}\Delta^6}$$

$$\text{Num.} = \frac{68[f_{-2} + f_{-1} + f_0 + f_1 + f_2]}{5}\Delta^6$$
$$- \frac{4[(-2)^2 f_{-2} + (-1)^2 f_{-1} + 0^2 f_0 + 1^2 f_1 + 2^2 f_2]\Delta^2}{5}\Delta^4$$
$$= \Delta^6 \left(\left(\frac{68}{25} - \frac{16}{5}\right)(f_{-2} + f_{+2}) + \left(\frac{68}{25} - \frac{4}{5}\right)(f_{-1} + f_{+1}) + \frac{68}{25} f_0 \right)$$

$$g_i = c$$
$$= \frac{1}{35} \begin{pmatrix} -3 & 12 & 17 & 12 & -3 \end{pmatrix} \begin{pmatrix} f_{i-2} \\ f_{i-1} \\ f_i \\ f_{i+1} \\ f_{i+2} \end{pmatrix}$$

- The weight at point i is maximum.
- The weights at both ends are negative.

$W(k)$



- $W(k)$ is flat in low frequency.

5.7 Gaussian Filter

Gaussian filter:

$\equiv$ Moving average with which the weight, $w(x')$, is a Gaussian function.

$$w(x) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{x^2}{2\sigma_x^2}} \quad \overset{\mathcal{F}}{\Longrightarrow} \quad W(k) = e^{-\frac{k^2}{2\sigma_k^2}} \quad \left(\sigma_k = \frac{1}{\sigma_x}\right)$$

(proof is shown in the next page.)

- $W(k)$ is a simple decreasing function.
- $W(k) > 0 \quad \rightarrow \quad$ No spurious resolution.

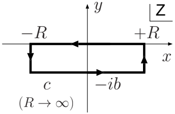
Fourier Transform of Gaussian function

$$
\begin{aligned}
W(k) &= \mathcal{F}\{w(x)\} \\
&= \frac{1}{\sqrt{2\pi}\sigma_x}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma_x^2}} e^{-ikx}\,dx \\
&\quad (X = \tfrac{x}{\sqrt{2}\sigma_x}) \\
&= \frac{1}{\sqrt{\pi}} e^{-\frac{\sigma_x^2}{2}k^2} \underbrace{\int_{-\infty}^{\infty} e^{-(X+i\frac{k\sigma_x}{\sqrt{2}})^2}\,dX}_{=\int e^{-X^2}\,dX=\sqrt{\pi} \quad (*)} \\
&= e^{-\frac{\sigma_x^2}{2}k^2} = e^{-\frac{1}{2\sigma_k^2}k^2} \qquad (\sigma_k = \tfrac{1}{\sigma_x})
\end{aligned}
$$

$$
\boxed{\mathcal{F}\left\{e^{-\frac{x^2}{2\sigma_x^2}}\right\} \propto e^{-\frac{\sigma_x^2 k^2}{2}}}
$$

(*)

$$
\begin{aligned}
I &= \int_{-\infty}^{\infty} e^{-(x+ib)^2}\,dx \\
&= \int_{-\infty-ib}^{\infty-ib} e^{-z^2}\,dz
\end{aligned}
$$



(No poles)

$$
\int_c e^{-z^2}\,dz = 0 \quad \rightarrow I = \int_{-\infty}^{\infty} e^{-x^2}\,dx
$$

$$
\begin{aligned}
I^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy \\
&= \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\,d\theta\,dr \qquad (t = r^2) \\
&= -\pi\left[e^{-t}\right]_0^{\infty} = \pi \\
&\therefore I = \sqrt{\pi}
\end{aligned}
$$

5.8 Cummulative Average of muliple measurement

- N_k sets of measurement.

$$f_i^{(k)} = \widetilde{f}_i + n_i^{(k)}$$
$$(k \in \{1, \cdots, N_k\})$$

- Cummulative Ave.

$$\langle\!\langle y_i \rangle\!\rangle \equiv \frac{1}{N_k} \sum_{k=1}^{N_k} y_i^{(k)}$$

- Cummul. Ave. of f_i

$$g_i = \langle\!\langle f_i \rangle\!\rangle = \widetilde{f}_i + \langle\!\langle n_i \rangle\!\rangle$$

- Expected value : $\mathrm{E}[g_i]$

$$\mathrm{E}[g_i] = \widetilde{f}_i + \underbrace{\mathrm{E}[\langle\!\langle n_i \rangle\!\rangle]}_{=\langle\!\langle \mathrm{E}[n_i] \rangle\!\rangle = 0}$$
$$= \widetilde{f}_i$$

- Variance of g_i : $\sigma_{g_i}^2$ (※ $\neq$ Var. of f_i)

$$\sigma_{g_i}^2 = \mathrm{E}\left[(g_i - \mathrm{E}[g_i])^2\right] = \mathrm{E}\left[\left\{(\widetilde{f}_i + \langle\!\langle n_i \rangle\!\rangle) - \widetilde{f}_i\right\}^2\right]$$
$$= \mathrm{E}\left[\langle\!\langle n_i \rangle\!\rangle^2\right] = \mathrm{E}\left[\frac{1}{N_k^2} \sum_k \sum_{k'} n_i^{(k)} n_i^{(k')}\right]$$
$$= \mathrm{E}\left[\frac{1}{N_k^2} \sum_k \left(\left(n_i^{(k)}\right)^2 + \sum_{k' \neq k} n_i^{(k)} n_i^{(k')}\right)\right]$$
$$= \frac{\sum_k \overbrace{\mathrm{E}\left[\left(n_i^{(k)}\right)^2\right]}^{\sigma_n^2}}{N_k^2} + \frac{\sum_k \sum_{k' \neq k} \overbrace{\mathrm{E}\left[n_i^{(k)} n_i^{(k')}\right]}^{0}}{N_k^2}$$
$$= \frac{1}{N_k^2} \sum_{k=1}^{N_k} \sigma_n^2 = \frac{\sigma_n^2}{N_k}$$

$\sqrt{\sigma_{g_i}^2}$ is called standard error.

Comparison of Cummulative Ave. and Moving Ave.

	Original	Moving Average	Cummulative Average
Ave.	$\widetilde{f_i}$	$\left\langle \widetilde{f_i} \right\rangle_m$	$\widetilde{f_i}$
Var.*	σ_n^2	$\dfrac{\sigma_n^2}{N_m}$	$\dfrac{\sigma_n^2}{N_k}$

(
* for White Noise
N_m : Number of averaging samples
N_k : Number of times of measurement
)

- The expected value of average is distorted by moving average, but not distorted by cummulative average.
- The variances become smaller for both averaging.
- If we can obtain measurements under same condition, cummlative average is superior.

5.9 Propagation of Error

Two independent measurements, f and g ($|df| \ll |\widetilde{f}|$, $|dg| \ll |\widetilde{g}|$) :

$$f = \widetilde{f} + df, \quad \mathrm{E}[f] = \widetilde{f}, \quad \mathrm{E}[df] = 0, \quad \mathrm{E}\left[(df)^2\right] = \sigma_f^2$$
$$g = \widetilde{g} + dg, \quad \mathrm{E}[g] = \widetilde{g}, \quad \mathrm{E}[dg] = 0, \quad \mathrm{E}\left[(dg)^2\right] = \sigma_g^2$$

Consider evaluetion of a new result : $h \equiv h(f,g) = \widetilde{h}(f,g) + dh(f,g)$

- Average:

$$dh = \left.\frac{\partial h}{\partial f}\right|_{(\widetilde{f},\widetilde{g})} df + \left.\frac{\partial h}{\partial g}\right|_{(\widetilde{f},\widetilde{g})} dg$$
$$= h'_f df + h'_g dg$$
$$\mathrm{E}[dh] = h'_f \mathrm{E}[df] + h'_g \mathrm{E}[dg]$$
$$= 0$$
$$\mathrm{E}[h] = \mathrm{E}\left[\widetilde{h} + dh\right] = \widetilde{h}(\widetilde{f},\widetilde{g})$$

- Variance:

$$\sigma_h^2 = \mathrm{E}\left[(dh)^2\right] = \mathrm{E}\left[\left(h'_f df + h'_g dg\right)^2\right]$$
$$= {h'_f}^2 \underbrace{\mathrm{E}\left[df^2\right]}_{\sigma_f^2} + {h'_g}^2 \underbrace{\mathrm{E}\left[dg^2\right]}_{\sigma_g^2} + 2\, h'_f h'_g \underbrace{\mathrm{E}[df \cdot dg]}_{=0}$$
$$= \left(\left.\frac{\partial h}{\partial f}\right|_{\widetilde{h}}\right)^2 \sigma_f^2 + \left(\left.\frac{\partial h}{\partial g}\right|_{\widetilde{h}}\right)^2 \sigma_g^2$$

Example of Error Propagation

$$h \equiv h(f,g)$$
$$\sigma_h^2 = \left(\frac{\partial h}{\partial f}\bigg|_{\widetilde{h}} \right)^2 \sigma_f^2 + \left(\frac{\partial h}{\partial g}\bigg|_{\widetilde{h}} \right)^2 \sigma_g^2$$

- Add. and Sub.
  - $h = f + g$
    $$\sigma_h^2 = \sigma_{f+g}^2 = \sigma_f^2 + \sigma_g^2$$
  - $h = f - g$
    $$\sigma_h^2 = \sigma_{f-g}^2 = \sigma_f^2 + \sigma_g^2$$

  Sum of each Variance.

- Mul. and Div.
  - $h = f \cdot g$
    $$\sigma_h^2 = \sigma_{f \cdot g}^2 = g^2\sigma_f^2 + f^2\sigma_g^2$$
    $$\frac{\sigma_h^2}{h^2} = \frac{\sigma_f^2}{f^2} + \frac{\sigma_g^2}{g^2}$$
  - $h = f/g$
    $$\sigma_h^2 = \sigma_{f/g}^2 = \frac{1}{g^2}\sigma_f^2 + \frac{f^2}{g^4}\sigma_g^2$$
    $$\frac{\sigma_h^2}{h^2} = \frac{\sigma_f^2}{f^2} + \frac{\sigma_g^2}{g^2}$$

  Sum of each normalized Variance.

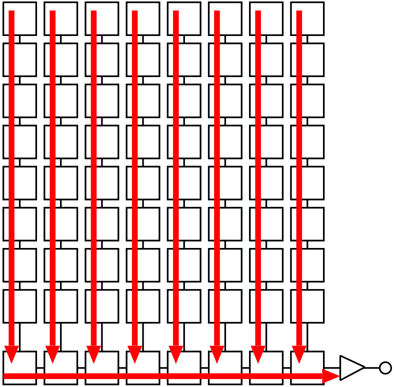
6. Image data
6.1 Image sensor

- Image Sensor
  - CCD (Charge Coupled Device)
  - CMOS (Complementary MOS(Metal Oxide Semiconductor))

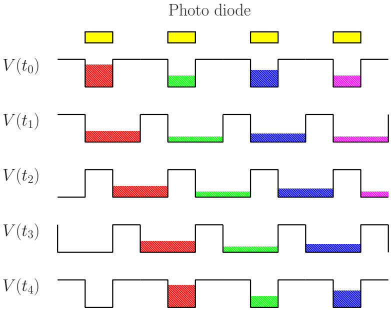
	CCD	CMOS
detection	Photo diode (Photo electron emission/excitation)	
Sensitivity	High	Low(High recently)
Amplifier, A/D converter	one for whole pixels	one for each pixel
Readout	Bucket relay (only whole pixels reading)	Addressing (possible to read one pixel)
Defect pixel	None	Existing

CCD

Circuit

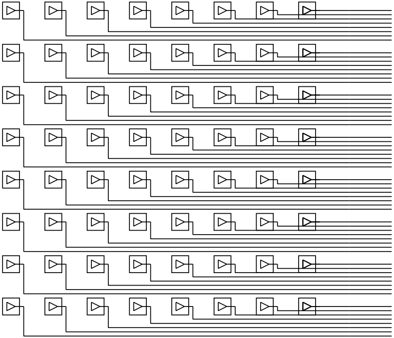


Bucket Relay



CMOS

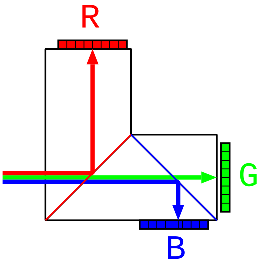
Circuit



Color Camera

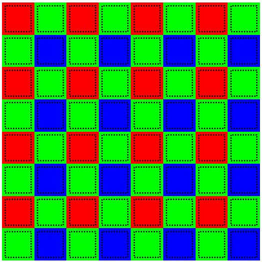
- three chips (high quality)

  Separating color by using prisms, each of which can reflect a certain color. Separated beams are detected each sensor.



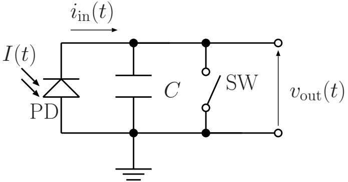
- one chip (small size)

  Color filter is placed in front of sensor.



6.2 Signal to Noise ratio and Resolution

- Charge accumuration by photo diode



- Incident Light $I(t) \propto i_{\mathrm{in}}(t)$
  - SW=ON(close) : $v_{\mathrm{out}}(t) = 0$
  - SW=OFF(Open) $(t = [0, T])$

    $v_{\mathrm{out}}(T) = \frac{1}{C} \int_0^T i_{\mathrm{in}}(t)\, dt$

    If $I(t) = \overline{I}$ (const.),

    $v_{\mathrm{out}}(T) = k\overline{I}T$.

    Signal is proportional to T.

- I includes fluctuation :

  $I(t) = \overline{I} + \delta I(t) \to \overline{I} \pm \sigma_{\delta I}$

  $\delta v_{\mathrm{out}}(T) = k \int_0^T \delta I(t)\, dt$

  $\frac{1}{T} \int_0^T \delta I(t)\, dt = \langle\!\langle \delta I \rangle\!\rangle$

  $\equiv$ Cummul. ave. of $\delta I(t)$

  $\sigma_{\langle\!\langle \delta I \rangle\!\rangle} \propto \sigma_{\delta I}/\sqrt{T}$ (δI is white)

  $\sigma_{v_{\mathrm{out}}}(T) = T\sigma_{\langle\!\langle \delta I \rangle\!\rangle} = k'\sigma_{\delta I}\sqrt{T}$

  Noise is proportional to $\sqrt{T}$.

Signal to Noise Ratio S/N :

$S/N \propto \sqrt{T}$

The quality of signal increases with increasing T.

- Area of Pixel A

  $i_{\text{in}}(t) \propto \int_A I(t,x,y)\,dA$

  - $I(t,x,y) = \overline{I}$:

    $v_{\text{out}}(A) = k\overline{I}A$

    v_{out} is proportional to A.

  - $I(t,x,y) = \overline{I} + \delta I(t,x,y)$:

    $\delta v_{\text{out}} = k\int_A \delta I(t,x,y)\,dA$

    $\sigma_{\langle\langle \delta I \rangle\rangle} \propto \sigma_{\delta I}/\sqrt{A}$

    $\sigma_{v_{\text{out}}}(A) = k'\sigma_{\delta I}\sqrt{A}$

    v_{out} is proportional to $\sqrt{A}$.

> Signal to Noise Ratio S/N :
>
> $S/N \propto \sqrt{A}$
>
> The quality of signal increases with increasing pixel size A.

- Resolution
  - Temporal resolution ⇔Exposure time
  - Spatial resolution ⇔Pixel Size

	S/N Larger is better	Resolution Smaller is better
time	$\propto \sqrt{T}$	T
space	$\propto \sqrt{A}$	$\sqrt{A}$

6.3 Discretization and Quantization

- Discretization and Quantization
  Continuous function $f(x)$
  $(x \in \mathbb{R},\ f(x) \in \mathbb{R})$
  - Discretization:
    (Digitizing Domain)
    $x_n = n\Delta x,\ n \in \mathbb{Z}$
  - Quantization:
    (Digitizing Range of f)
    $f_m = m\Delta f,\ m \in \mathbb{Z}$

- Image Data
  - Pixel : Discrete point
    $\Leftrightarrow \quad i, j \in \mathbb{Z}$
    (e.g. 640×400, 1024×768)
  - Intensity : Quantized of brightness
    $\Leftrightarrow \quad I_{i,j} \in \mathbb{Z}$
    A/D (Analog to Digital) converter

    e.g.

8bits	(0,···, 255)
10bits	(0,···, 1023)
12bits	(0,···, 4095)

6.4 Correction of Intensity

γ correction

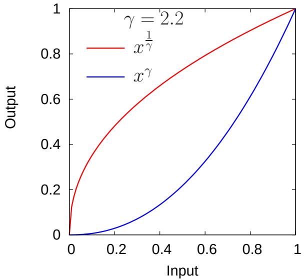
$$I_i \overset{\text{Record}}{\Longrightarrow} f_r \overset{\text{Display}}{\Longrightarrow} I_o$$

- Display device
  Gain of CRT is not linear
  $\rightarrow I_o \not\propto f_r$
  $I_o \propto f_r^{\gamma_d}$ (e.g. $\gamma_d = 2.2$)
- Recording device
  $f_r \propto I_i^{\gamma_r}$

$$\therefore I_o = I_i^{\gamma_r \cdot \gamma_d}$$

- In general, recording device has $\gamma_r = 1/\gamma_d$ so that $I_o = I_i$.
- This is not appropriate to quantitative processing.
- If we wish quantitative evaluation, apply the cancelation of the γ correction

$$\widehat{f} = f_r^{1/\gamma_r} \propto I_i.$$



6.5 Image format

- Single format

Format	Name	Color	bit	Compres.	Revers.	Multi-image
PBM	Portable Bit Map	White/Black	1	×	○	×
PGM	Portable Gray Map	Gray	8	×	○	×
PPM	Portable Pixel Map	RGB	3×8	×	○	×
GIF	Graphics Interchange Format	RGB	3×8	○	○	○
JPEG	Joint Photographic Experts Group	RGB	3×8	○	×	×
PNG	Portable Network Graphics	RGB-alpha*	4×16	○	○	×

* : alpha is a channel for transparency

- Integrated Multiple formats

Format	Name
PNM	Portable aNy Map (PBM, PGM, PPM)
TIFF	Tagged Image File Format
BMP	Microsoft windows BitMaP

7. Simple filter for image using convolution
7.1 Smoothing using Moving Average

$$g_{i,j} = \sum_{m,n} w_{m,n} f_{i+m,j+n} \qquad m, n \in -N, \ldots, N, \quad N_m = 2N + 1$$

size	Simple ave.	Square pyramid	circular	Gaussian like
3×3 $\begin{pmatrix} N_m = 3 \\ N = 1 \end{pmatrix}$	$\begin{array}{\|c\|c\|c\|}\hline 1&1&1\\ \hline 1&1&1\\ \hline 1&1&1\\ \hline \end{array} \times 1/9$	$\begin{array}{\|c\|c\|c\|}\hline 1&1&1\\ \hline 1&2&1\\ \hline 1&1&1\\ \hline \end{array} \times 1/10$	$\begin{array}{\|c\|c\|c\|}\hline 0&1&0\\ \hline 1&1&1\\ \hline 0&1&0\\ \hline \end{array} \times 1/5$	$\begin{array}{\|c\|c\|c\|}\hline 1&2&1\\ \hline 2&4&2\\ \hline 1&2&1\\ \hline \end{array} \times 1/16$
5×5 $\begin{pmatrix} N_m = 5 \\ N = 2 \end{pmatrix}$	$\begin{array}{\|c\|c\|c\|c\|c\|}\hline 1&1&1&1&1\\ \hline 1&1&1&1&1\\ \hline 1&1&1&1&1\\ \hline 1&1&1&1&1\\ \hline 1&1&1&1&1\\ \hline \end{array} \times 1/25$	$\begin{array}{\|c\|c\|c\|c\|c\|}\hline 1&1&1&1&1\\ \hline 1&2&2&2&1\\ \hline 1&2&3&2&1\\ \hline 1&2&2&2&1\\ \hline 1&1&1&1&1\\ \hline \end{array} \times 1/35$	$\begin{array}{\|c\|c\|c\|c\|c\|}\hline 0&0&1&0&0\\ \hline 0&1&1&1&0\\ \hline 1&1&1&1&1\\ \hline 0&1&1&1&0\\ \hline 0&0&1&0&0\\ \hline \end{array} \times 1/13$	$\begin{array}{\|c\|c\|c\|c\|c\|}\hline 1&4&6&4&1\\ \hline 4&16&24&16&4\\ \hline 6&24&36&24&6\\ \hline 4&16&24&16&4\\ \hline 1&4&6&4&1\\ \hline \end{array} \times 1/256$

7.2 Smoothing using Non-linear Filter

Since the moving average losses higher frequency component, edges and corners in the original image are blurred.

$f_{i+m,j+n}$

0	0	9	9	9
0	0	9	9	9
0	0	9	9	9
0	0	18	18	18
0	0	18	18	18

$*$

$w_{m,n}$

1	1	1
1	1	1
1	1	1

$=$

$g_{i,j}$

0	3	6	9	9
0	3	6	9	9
0	4	8	12	12
0	5	10	15	15
0	6	12	18	18

- Median filter
  Reducible the blurring edges
- Adaptive local averaging filter
  Reducible the blurring corners

Note:

These filters are irreversible.

Median Filter

Replace the pixel value at the central pixel in partial region by the median in the region.

$f_{i,j}$

①	③	②	3	1
④	⑤	①	9	5
⑥	⑤	⑧	6	6
1	2	7	[5]	[7]
1	2	8	[4]	[4]

sort and select median

○ : 1, 1, 2, 3, 4, 5, 5, 6, 8 (Median: 4)

□ : 4, 4, 5, 7
Median: $\frac{4+5}{2} = 4.5$

⇓

$g_{i,j}$

*	*	*	*	*
*	4	*	*	*
*	*	*	*	*
*	*	*	*	*
*	*	*	*	4.5

⇐

$g_{i,j}$

3.5	2.5	4	2.5	4
4.5	4	5	5	5.5
4.5	5	5	6	6
2	5	5	6	5.5
1.5	2	4.5	6	4.5

Reduction of blurring of edges.

Input:

0	0	9	9	9
0	0	9	9	9
0	0	9	9	9
0	0	18	18	18
0	0	18	18	18

Moving Average

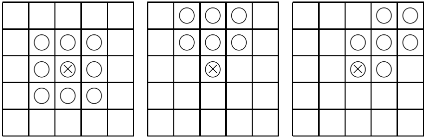
0	3	6	9	9
0	3	6	9	9
0	4	8	12	12
0	5	10	15	15
0	6	12	18	18

Median Filter

0	0	9	9	9
0	0	9	9	9
0	0	9	9	9
0	0	9	18	18
0	0	18	18	18

Adaptive Local Averaging Filter

Sampling patterns:



9 points (Symmetric(×1)) (C) — 7 points (× 4 patterns) (N,W,S,E) — 7 points (× 4 patterns) (NE,NW,SW,SE)

- Evaluate variance for each pattern.

$$\sigma^2_{(p)} \qquad (p \in \{\text{C, N,W,S,E, NE,NW,SW,SE}\})$$

- $g_{i,j}$ is taken as the average $\overline{f^{(p)}_{i,j}}$ with minimum variance $\sigma^2_{(p)}$.

Reduction of blurring of edges.

Input

0	0	9	9	9
0	0	9	9	9
0	0	9	9	9
0	0	18	18	18
0	0	18	18	18

Moving Average

0	3	6	9	9
0	3	6	9	9
0	4	8	12	12
0	5	10	15	15
0	6	12	18	18

Median Filter

0	0	9	9	9
0	0	9	9	9
0	0	9	9	9
0	0	9	18	18
0	0	18	18	18

Adapt. Local Ave.

0	0	9	9	9
0	0	9	9	9
0	0	9	9	9
0	0	18	18	18
0	0	18	18	18

7.3 Remove defect pixel

- Defect pixel : Pixels with $f_{i,j} = 0$ or $f_{i,j} = 255$(Max.)
- Reason : Difference of gain between pixels.
- Apply the Median filter to only the defect pixels.

Original | Defects | Median for whole image | Median for defect pixel

defects | removed smoothed | removed Not smoothed

7.4 Edge detection

Edge: Points having local maximum of the gradient ($|\nabla f|$).

Edge detection by gradient

- Derivative

$$\frac{df}{dx} = \lim_{\Delta \to 0} \frac{f(x+\frac{\Delta}{2}) - f(x-\frac{\Delta}{2})}{\Delta} = \lim_{\Delta \to 0} \frac{f(x+\Delta) - f(x-\Delta)}{2\Delta}$$

- Difference($\Delta \neq 0$)

$$\frac{\Delta f}{\Delta x} = \frac{f(x+\Delta) - f(x-\Delta)}{2\Delta} = \frac{1}{2}(f_{i+1} - f_{i-1})$$

$$\frac{df}{dx} \simeq \frac{\Delta f}{\Delta x}$$

$$g_{i,j} = \sum_{m,n} w_{m,n} f_{i+m,j+n} \quad \begin{pmatrix} m,n \in -N,\ldots,N \\ N_m = 2N+1 \end{pmatrix}$$

Example of $w_{m,n}$

- $\nabla f \cdot \boldsymbol{e}_x$

0	0	0
$\frac{-1}{2}$	0	$\frac{1}{2}$
0	0	0

- $\nabla f \cdot \boldsymbol{e}_y$

0	$\frac{1}{2}$	0
0	0	0
0	$\frac{-1}{2}$	0

- $\nabla f \cdot (\boldsymbol{e}_x + \boldsymbol{e}_y)$

0	0	$\frac{1}{2\sqrt{2}}$
0	0	0
$\frac{-1}{2\sqrt{2}}$	0	0

- $\nabla f \cdot (\boldsymbol{e}_x - \boldsymbol{e}_y)$

$\frac{-1}{2\sqrt{2}}$	0	0
0	0	0
0	0	$\frac{1}{2\sqrt{2}}$

Example of Edge detection

Original

$\nabla f \cdot \boldsymbol{e}_x$

$\nabla f \cdot \boldsymbol{e}_y$

$|\nabla f|$

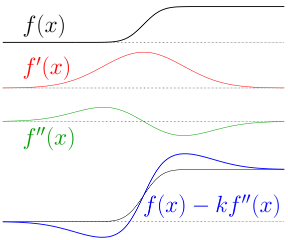
- $\nabla f \cdot \boldsymbol{e}_x$, $\nabla f \cdot \boldsymbol{e}_y$ embosses image.
- By $\nabla f \cdot \boldsymbol{e}_x$, the vertical edge can be detected, but horizontal one cannot.
- $|\nabla f|$ is useful for edge detection.

$|\nabla f|$ (inverted)

7.5 Enhancement of Edge

Using Laplacian

Consider curvature ($f''(x)$)



Enhancement of Edge :

$$g(\boldsymbol{r}) = f(\boldsymbol{r}) - k\nabla^2 f(\boldsymbol{r})$$

Second derivative

- 1-dim. $f''_i = f'_{i+1/2} - f'_{i-1/2}$
  $= f_{i+1} + f_{i-1} - 2f_i$
- 2-dim. Laplacian($\nabla^2 f$)
  $f''_{i,j} = f_{i+1,j} + f_{i-1,j} + f_{i,j+1} + f_{i,j-1} - 4f_{i,j}$

In the case $k = 1$,

f

0	0	0
0	1	0
0	0	0

$-$ $(\nabla^2 f)$

0	1	0
1	−4	1
0	1	0

$=$ $f - (\nabla^2 f)$

0	−1	0
−1	5	−1
0	−1	0

Example of Edge Enhancement(using Laplacian)

Smoothed

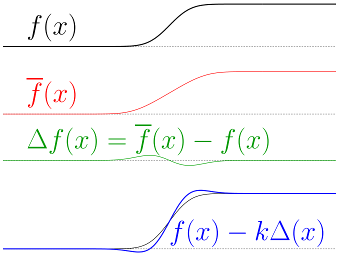
Enhance$(k = 2)$

Enhance$(k = 4)$

Original

Using diff. from Smoothed

Consider diff. from smoothed one $(\overline{f}(x) - f(x))$



Enhancement of Edge :

$$g(\boldsymbol{r}) = f(\boldsymbol{r}) - k\left(\overline{f(\boldsymbol{r})} - f(\boldsymbol{r})\right)$$
$$= (1+k)f(\boldsymbol{r}) - k\overline{f(\boldsymbol{r})}$$

In the case $k = 1$ and 3x3 simple ave.,

$$g = 2f - \overline{f}$$

$2f$

0	0	0
0	2	0
0	0	0

$-$ $\overline{f}$

$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$
$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$
$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$

$=$ $2f - \overline{f}$

$\frac{-1}{9}$	$\frac{-1}{9}$	$\frac{-1}{9}$
$\frac{-1}{9}$	$\frac{17}{9}$	$\frac{-1}{9}$
$\frac{-1}{9}$	$\frac{-1}{9}$	$\frac{-1}{9}$

Example of Edge Enhancement(using diff. from smoothed)

Smoothed

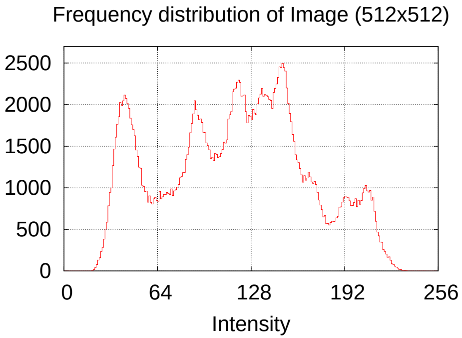
Enhance($k = 2$)

Enhance($k = 4$)

Original

7.6 Frequency distribution (Histogram)

Histogram $\rightarrow$ Used for binarizing or labeling



In this example the thresholds are about 64, 100, 126, 180.

7.7 Binarizing

Binarizing : Distinguish either true or false of a condition

1: True, 0: False

- Comparison with threshold

  e.g. $f(x,y) \geq f_{\rm th}$
- Solutions (Points) of equations

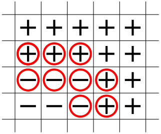
  e.g. $f(x,y) = a$

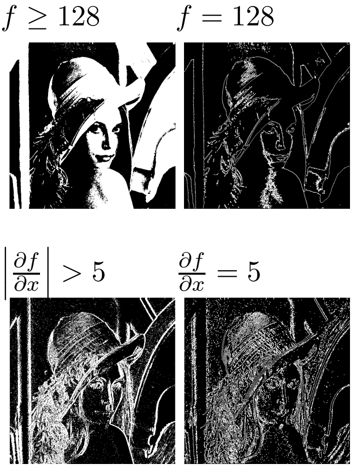
<u>Solution of $f(x,y) = a$</u>

$$(f(x,y) - a) \cdot (f(x',y') - a) \leq 0$$

($(x',y')$is adjacent pixels of (x,y))

In the case of 4 direction search:





7.8 Partitioning

Partitioning by Edge

Edge: Points with local max. of $|\nabla f|$.

$\rightarrow$ Solution of $\nabla^2 f = 0$?

$$\left(\begin{array}{l} \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \\ \quad \text{Even if } \frac{\partial^2 f}{\partial x^2} \neq 0 \text{ and } \frac{\partial^2 f}{\partial y^2} \neq 0 \text{ ,} \\ \quad \text{in the case of } \frac{\partial^2 f}{\partial x^2} = -\frac{\partial^2 f}{\partial y^2}, \\ \quad \nabla^2 f = 0. \quad \rightarrow \times \end{array} \right)$$

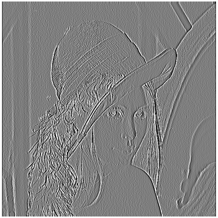
$$\therefore \boldsymbol{r} = \left\{ (x, y) \middle| \frac{\partial^2 f}{\partial x^2} = 0,\ \frac{\partial^2 f}{\partial y^2} = 0 \right\}$$

However, this is also inappropriate.

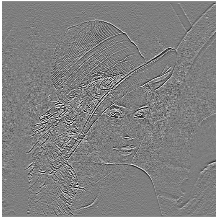
$\because$ The second derivatives have high frequency component, there are many zeros.

(Max: White, Min: Black)

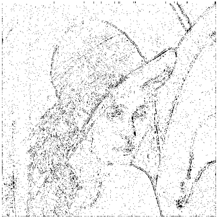
$\frac{\partial^2 f}{\partial x^2}$ [-20,20]



$\frac{\partial^2 f}{\partial y^2}$ [-20,20]



$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = 0$



$\begin{pmatrix} = 0 : \text{White} \\ \neq 0 : \text{Black} \end{pmatrix}$

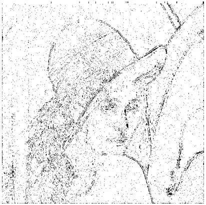
Improvement of the solution search algorithm

Methods to find the sol. of $\frac{\partial^2 f}{\partial x^2} = 0$.

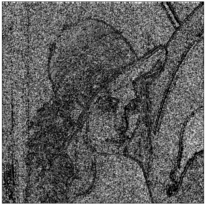
- Simple search
  $f''_i \cdot f''_{i+1} < 0$ or $f''_i \cdot f''_{i-1} \leq 0$
  $\rightarrow$ Broaden

- Use of average of half shifted points.
  $f''_{i+1/2} \cdot f''_{i-1/2} \leq 0 \rightarrow f''_i = 0$
  $\left(f''_{i\pm 1/2} = \frac{f''_i + f''_{i\pm 1}}{2} \right)$
  $\rightarrow$ This can avoid broadening.

Simple method



Ave. of Half shifted pix.



(White:Solution)

Solution is improved. However, it remains many solutions because of higher freq. components.
$\rightarrow$Coupling method with other methods is required.

Edge detection by $|\nabla f|$

Edge: Points having local maximum of the gradient ($|\nabla f|$).

$\Longrightarrow$ Points having large gradient.

Problem :

- Appropriate threshold

  $$|\nabla f| > |\nabla f|_0$$

- Broaden edges
  - Use with the second derivative.
  - Use of variance of gradient.

Variance of gradient

$$\overline{\nabla f} = (\overline{f'_x}, \overline{f'_y})$$

$$\sigma^2_{\nabla f} = \sigma^2_{f'_x} + \sigma^2_{f'_y}, \quad \widehat{\sigma}_{\nabla f} = \frac{\sigma_{\nabla f}}{|\nabla f|}$$

Condition of edge:

- $|\nabla f| > |\nabla f|_0$
- $\widehat{\sigma}_{\nabla f} < \widehat{\sigma}_{\nabla f\,0}$
- $\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = 0$

Example of Edge detection by $|\nabla f|$

$|\nabla f| \geq 8$,
$f_{'x'x} = f_{'y'y} = 0$

$|\nabla f| \geq 8$,
$\sigma_{\nabla f} / |\nabla f| \leq 0.7$

$\sigma_{\nabla f} / |\nabla f| \leq 0.7$,
$f_{'x'x} = f_{'y'y} = 0$

$|\nabla f| \geq 8$,
$\sigma_{\nabla f} / |\nabla f| \leq 0.7$,
$f_{'x'x} = f_{'y'y} = 0$

$|\nabla f| \geq 8$,
$\sigma_{\nabla f} / |\nabla f| \leq 0.7$,
$f_{'x'x} = f_{'y'y} = 0$

Disconnected edge

Expansion and Contraction of Binary image

- Expansion

$$g_{i,j} = \begin{cases} 1 & \text{One of neighbors has 1.} \\ 0 & \text{otherwise} \end{cases}$$

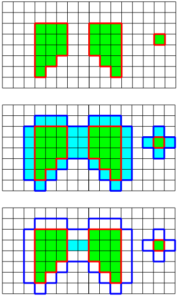
Connect regions

- Contraction

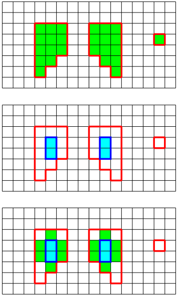
$$g_{i,j} = \begin{cases} 0 & \text{One of neighbors has 0.} \\ 1 & \text{otherwise} \end{cases}$$

Disconnect a region,
Remove isolated point.

Original
⇓
Expansion
⇓
Contraction

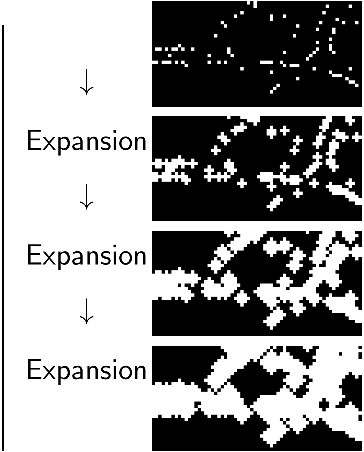


Original
⇓
Contraction
⇓
Expansion



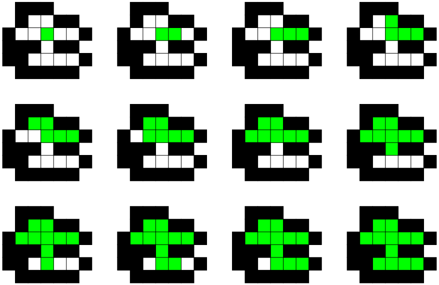
Expansion$\neq$ Inv{Contraction}

Connection of Edges by Expansion



7.9 Painting (Labeling)

Until finding border set mark



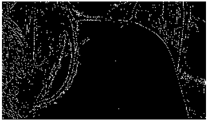
```
begin function flood_fill(i, j)
   if flag L_{i,j} is not marked then
      set flag L_{i,j} = D as internal
      call flood_fill(i+1, j  )
      call flood_fill(i  , j+1)
      call flood_fill(i-1, j  )
      call flood_fill(i  , j-1)
   end if
end function

begin main
   set flag L_{i,j} = B for boundary
   select initial point (i_0, j_0)
   flood_fill(i_0, j_0)
end main
```

Recursive coding

Example of Painting

Original

Edge detection



Connect broken edge
(Exp edge 3times)

Flood fill

Widen region
(Expand region 3times)

Paint isolated
(Expand region 3times
Contract reg. 3times)

merged
(with detected Edge)

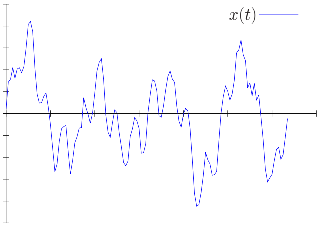
This method requires several try to tune parameters.
Unfortunately, there are no automatic methods.

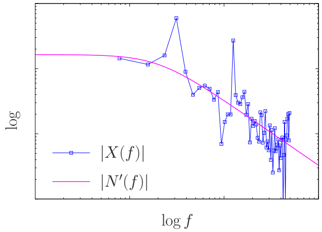
8. Noise Reduction using Spectrum
8.1 Wiener filter

Model of Observation : $$x(t) = \widehat{x}(t) + n(t) \qquad (1)$$

$x(t)$: Observed (known)	$X(f)$ (known)	$\lvert X(f)\rvert$ (known)	
$\widehat{x}(t)$: True (Unknown)	$\widehat{X}(f)$ (Unknown)	$\lvert\widehat{X}(f)\rvert$ (Unknown)	
$n(t)$: Noise (Unknown)	$N(f)$ (Unknown)	$\lvert N(f)\rvert$ (unknown), $\lvert N'(f)\rvert$ (known)	

$\left(\lvert N'(f)\rvert$: sustitute of $\lvert N(f)\rvert$, white/red, $\sigma^2_{n'}, \ldots \right)$





$\downarrow$

$\widehat{x}(t)$?

Parseval's theorem

Parseval's theorem

$$\int_{-\infty}^{\infty} |a(t)|^2 \, dt = \int_{-\infty}^{\infty} |A(f)|^2 \, df \quad (2)$$

$$\begin{cases} A(f) = \int_{-\infty}^{\infty} a(t) e^{-j2\pi f t} \, dt \\ a(t) \;\; = \int_{-\infty}^{\infty} A(f) e^{+j2\pi f t} \, df \end{cases} \quad (3)$$

(Proof)

$$\begin{aligned} \text{LHS} &= \int_t \left(\int_f A(f) e^{+j2\pi f t} \, df \cdot \int_{f'} A^*(f') e^{-j2\pi f' t} \, df' \right) dt \\ &= \int_f \int_{f'} A(f) A^*(f') \underbrace{\int_t e^{+j2\pi (f-f')t} \, dt}_{=\delta(f-f')} \, df' \, df \\ &= \int_f \int_{f'} A(f) A^*(f') \, \delta(f - f') \, df' \, df \\ &= \int_f A(f) A^*(f) \, df = \int_{-\infty}^{\infty} |A(f)|^2 \, df = \text{RHS} \end{aligned}$$

Spectral product of Non-Correlated Signals

If $a(t)$ is independent $n(t)$

$$\int A^*(f)N(f)\,df = C_{a,n}(0) = 0. \tag{4}$$

Integral of spectral product of Non-Correlated Signals vanishes.

Proof.

$$\int A^*(f)N(f)\,df \qquad \text{(Express } A(f) \text{ and } N(f) \text{ by FT)}$$

$$= \int_f \int_t a^*(t)e^{+j2\pi ft}\,dt \int_{t'} n(t')e^{-j2\pi ft'}\,dt'\,df \quad \text{(Exchange the order)}$$

$$= \int_t \int_{t'} a^*(t)n(t') \int_f e^{+j2\pi f(t-t')}\,df\,dt'\,dt \quad \left(\int e^{+j2\pi f(t-t')}\,df = \delta(t-t')\right)$$

$$= \int_t a^*(t) \int_{t'} n(t')\delta(t-t')\,dt'\,dt = \int_t a^*(t)n(t)\,dt = C_{a,n}(0) = 0.$$

Filtering function in Sprctral domain $\Phi_x(f)$

$$x(t) = \widehat{x}(t) + n(t) \quad (x, \widehat{x}, n \in \mathbb{R}) \quad (1)$$

$$X(f) = \widehat{X}(f) + N(f) \; (X, \widehat{X}, N \in \mathbb{C}) \quad (5)$$

$$\widetilde{X}(f) = X(f)\Phi_x(f) \quad (6)$$

$$\widetilde{x}(t) = \mathcal{F}^{-1}\left\{\widetilde{X}(f)\right\} \quad (7)$$

$\Phi_x(f) \in \mathbb{R}$: Filtering function
$\widetilde{X}(f) \in \mathbb{C}$: Estimated spectrum
$\widetilde{x}(t)$: Estimated spectrum

- Determine $\widetilde{x}(t)$ so that the residual is minimized.

(Integral of residual)

$$E = \int |\widetilde{x}(t) - \widehat{x}(t)|^2 \, dt \quad (8)$$

- From the Parseval's theorem

$$E = \int |\widetilde{x}(t) - \widehat{x}(t)|^2 \, dt$$
$$= \int \underbrace{\left|\widetilde{X}(f) - \widehat{X}(f)\right|^2}_{\equiv I(f) \geq 0} df \quad (9)$$

- Since integrand $I(f) \geq 0$,
  Minimize $E \Leftrightarrow$ Minimize $I(f)$ for all f.

$$\rightarrow \quad \frac{\partial E}{\partial \Phi_x} = 0 \quad (10)$$

(Stationary condition)

- $$E = \int \left|\widetilde{X} - \widehat{X}\right|^2 df = \int \left|X\Phi_x - \widehat{X}\right|^2 df$$
  E is the function of the function Φ_x
  This is called functional.

$$
\begin{aligned}
I &= \left|\widetilde{X} - \widehat{X}\right|^2 = \left|X\Phi_x - \widehat{X}\right|^2 \\
&= \left|(\widehat{X} + N)\Phi_x - \widehat{X}\right|^2 = \left|\widehat{X}(\Phi_x - 1) + N\Phi_x\right|^2 \\
&= (\widehat{X}(\Phi_x - 1) + N\Phi_x)^*(\widehat{X}(\Phi_x - 1) + N\Phi_x) \\
&= \underbrace{\left|\widehat{X}\right|^2 (\Phi_x - 1)^2 + |N|^2 \Phi_x^2}_{\equiv I'} \\
&\quad + \underbrace{(\widehat{X}^* N + \widehat{X} N^*)(\Phi_x - 1)\Phi_x}_{\text{Integral over } f \text{ vanishes.} (\because \text{ Eq.}(4))}
\end{aligned}
$$

$$
E = \int I \, df = \int I' \, df
$$

$$
I' = \left|\widehat{X}\right|^2 (\Phi_x - 1)^2 + |N|^2 \Phi_x^2 \geq 0
$$

minimize E $\Leftrightarrow$ minimize I' $\Leftrightarrow \dfrac{\partial I'}{\partial \Phi_x} = 0$

$$
\frac{\partial I'}{\partial \Phi_x} = 2\left(\left|\widehat{X}\right|^2 (\Phi_x - 1) + |N|^2 \Phi_x\right) = 0
$$

$$
\rightarrow \Phi_x = \frac{\left|\widehat{X}\right|^2}{\left|\widehat{X}\right|^2 + |N|^2}
$$

Wiener filter

$$
\Phi_x(f) = \frac{\left|\widehat{X}(f)\right|^2}{\left|\widehat{X}(f)\right|^2 + |N(f)|^2} \tag{11}
$$

$$
(0 \leq \Phi_x(f) \leq 1)
$$

However, this form includes FT of true solution, $\widehat{X}(f)$.

Representation of filter function using observed value

$$\begin{pmatrix} X = \widehat{X} + N \\ \widetilde{X} = X\Phi_x \end{pmatrix}$$

$$\begin{aligned}
I &= \left|\widetilde{X} - \widehat{X}\right|^2 = \left|X\Phi_x - (X - N)\right|^2 \\
&= \left|X(\Phi_x - 1) + N\right|^2 \\
&= (X(\Phi_x - 1) + N)^*(X(\Phi_x - 1) + N) \\
&= |X|^2 (\Phi_x - 1)^2 + |N|^2 \\
&\quad + (X^*N + XN^*)(\Phi_x - 1)
\end{aligned}$$

$$\begin{pmatrix} X^*N + XN^* \\ \quad = (\widehat{X} + N)^*N + (\widehat{X} + N)N^* \\ \quad = \widehat{X}^*N + \widehat{X}N^* + 2|N|^2 \end{pmatrix}$$

$$\begin{aligned}
&= |X|^2 (\Phi_x - 1)^2 - |N|^2 + 2|N|^2 \Phi_x \\
&\quad + (\widehat{X}^*N + \widehat{X}N^*)(\Phi_x - 1)
\end{aligned}$$

Since $\int A^*N\,df = 0$ (Eq.(4))
the last term is removed from I.

$$E = \int I\,df = \int I'\,df$$

$$I' = |X|^2 (\Phi_x - 1)^2 - |N|^2 + 2|N|^2 \Phi_x$$

$$\frac{\partial I'}{\partial \Phi_x} = 2\left(|X|^2 (\Phi_x - 1) + |N|^2\right) = 0$$

$$\rightarrow \quad \Phi_x = \frac{|X|^2 - |N|^2}{|X|^2} \simeq \frac{|X|^2 - |N'|^2}{|X|^2}$$

$$(\because N' \simeq N)$$

(All vars. in RHS are known.)

$\therefore$

$$\Phi_x(f) = \frac{|X(f)|^2 - |N'(f)|^2}{|X(f)|^2} \qquad (12)$$

Steps to Apply Wiener filter

1. $|X(f)|^2$
   1. Measurement of $x(t)$
   2. $|X(f)|^2 = |\mathcal{F}\{x(t)\}|^2$

2. $|N'(f)|^2$
   - Measurement of $n(t)$ (without signal) $|N'(f)|^2 = |\mathcal{F}\{n'(t)\}|^2$
   - If impossible, detemine considering property of noise.
     - White : $|N'(f)| = \text{const}$
     - Brownian : $|N'(f)| \propto \frac{1}{f^2+a^2}$

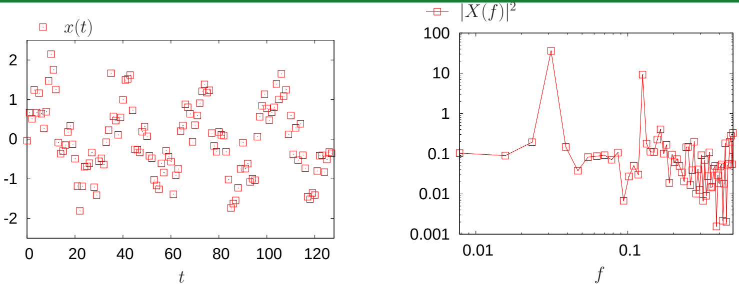
3. $\Phi_x(f) = \frac{|X(f)|^2 - |N'(f)|^2}{|X(f)|^2}$
   - If $\Phi_x(f) < 0$, $\Phi_x(f) = 0$. (irreversible)

   Wiener filter only corrects the amplitude, it does not correct phase.

   $(\Phi_x < 0 \Leftrightarrow |\Phi_x| e^{-i\pi})$

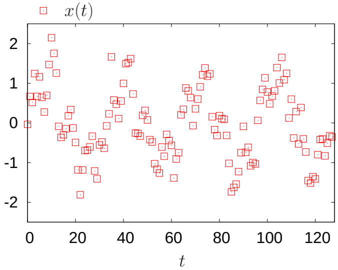
4. $\widetilde{x}(t)$
   1. $\widetilde{X}(f) = X(f)\Phi_x(f)$
   2. $\widetilde{x}(t) = \mathcal{F}^{-1}\left\{\widetilde{X}(f)\right\}$

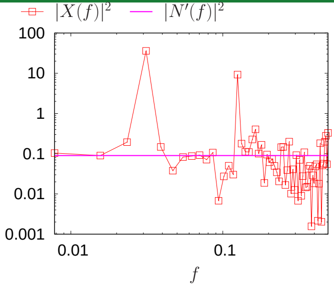
Example applying Wiener filter.



1. $x(t)$, $X(f)$

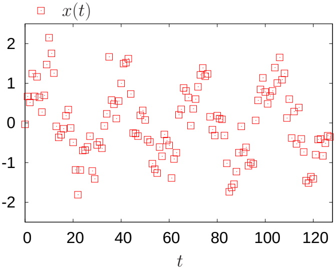
Example applying Wiener filter.

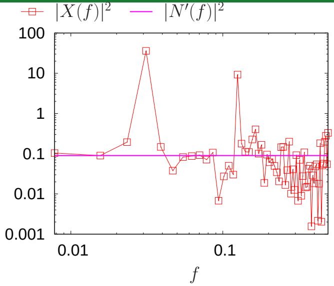




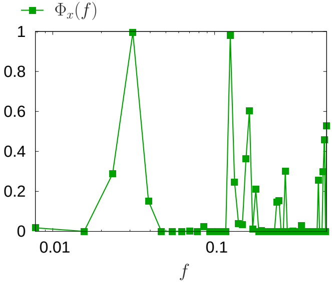
1. $x(t)$, $X(f)$
2. $N'(f)$

Example applying Wiener filter.

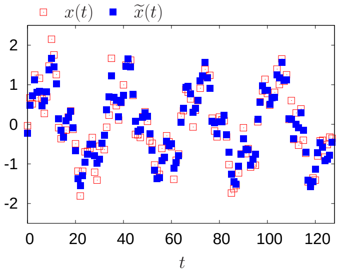


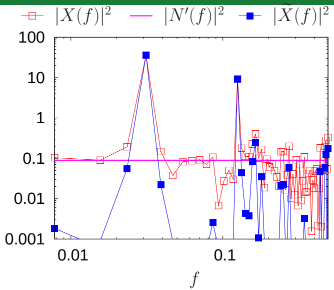


1. $x(t)$, $X(f)$
2. $N'(f)$
3. $\Phi_x(f) = \frac{|X(f)|^2 - |N'(f)|^2}{|X(f)|^2}$

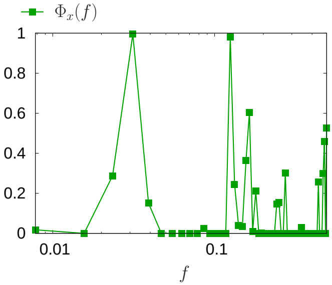


Example applying Wiener filter.





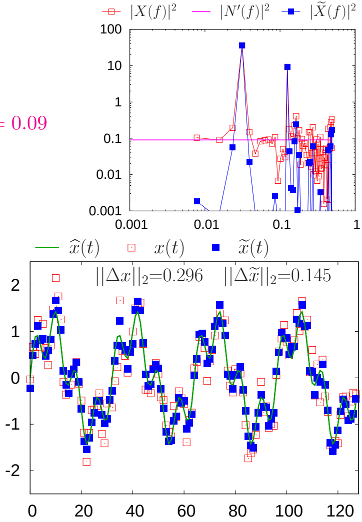
1. $x(t)$, $X(f)$
2. $N'(f)$
3. $\Phi_x(f) = \frac{|X(f)|^2 - |N'(f)|^2}{|X(f)|^2}$
4. $\widetilde{X}(f)$, $\widetilde{x}(t)$



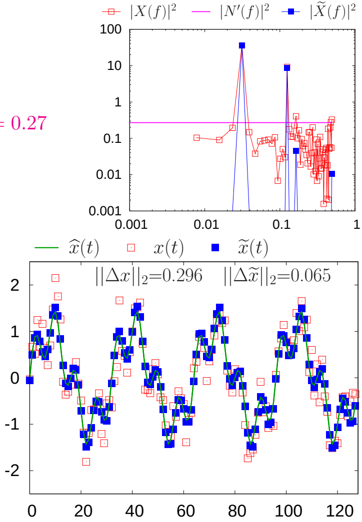
Effect of $|N'(f)|$ (White Noise)

$$\left(x(t) = \sin\left(\frac{2\pi t}{T/4}\right) + \frac{1}{2}\sin\left(\frac{2\pi t}{T/16}\right) + n, \quad \sigma_n^2 = 0.09\right)$$

$\sigma_n^2 = 0.09$



$\sigma_n^2 = 0.27$

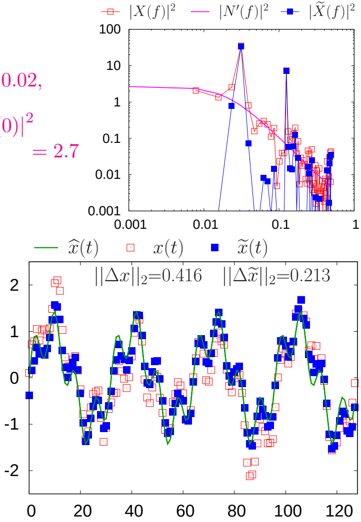


Effect of $|N'(f)|$ (Brownian Noise)

$$\left(x(t) = \sin\left(\frac{2\pi t}{T/4}\right) + \frac{1}{2}\sin\left(\frac{2\pi t}{T/16}\right) + r, \quad \alpha = 0.02, \quad \sigma_{n'}^2 = 0.04 \quad (|N'(0)|^2 = 2.7)\right)$$

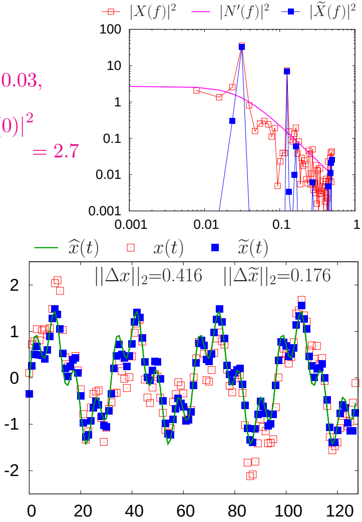
$\alpha = 0.02$,

$|N'(0)|^2 = 2.7$



$\alpha = 0.03$,

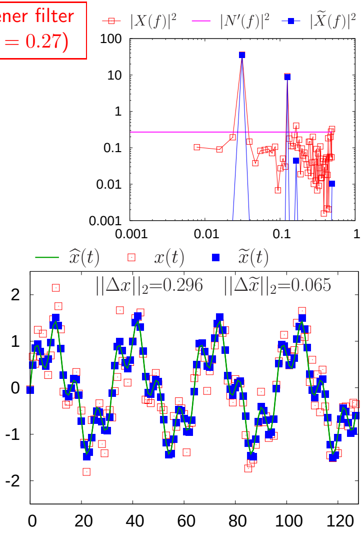
$|N'(0)|^2 = 2.7$



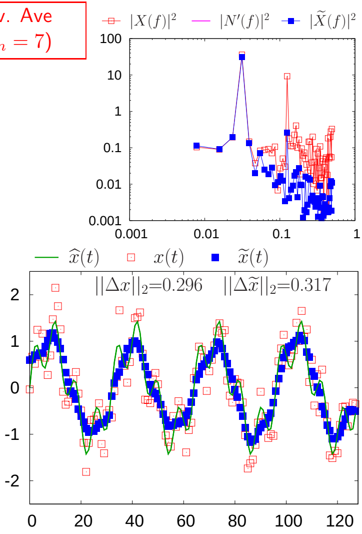
Comparison between Wiener filter and Moving average

$$\left(x(t) = \sin\left(\frac{2\pi t}{T/4}\right) + \frac{1}{2}\sin\left(\frac{2\pi t}{T/16}\right) + n, \quad \sigma_n^2 = 0.09\right)$$

Wiener filter ($\sigma_n^2 = 0.27$)



Mov. Ave ($N_m = 7$)



Summary of Wiener filter

In the case where $x(t)$ and $|N'(f)|$ is known:

$$X(f) = \mathcal{F}\{x(t)\}$$
$$\Phi_x(f) = \frac{|X(f)|^2 - |N'(f)|^2}{|X(f)|^2}$$
$$\widetilde{X}(f) = X(f)\Phi_x(f)$$
$$\widetilde{x}(t) = \mathcal{F}^{-1}\left\{\widetilde{X}(f)\right\}$$

- Wiener filter can be taken into account of Noise spectrum.
- Wiener filter is applicable when the spectrum of signal has several peaks. This is different from the moving average.

Wiener filter is called 'Optimal filter'.

8.2 Wiener deconvolution

- Model of Observation :

$$y(t) = h(t) * \widehat{x}(t) + n(t) \quad (1)$$

$$(y(t), h(t) : \text{known})$$

$$Y(f) = H(f) \cdot \widehat{X}(f) + N(f) \quad (2)$$

$$\left(\widehat{X} = \frac{Y-N}{H}\right)$$

$$|N(f)| \sim |N'(f)| \quad (|N'(f)| : \text{known}) \quad (3)$$

- Estimation $\widetilde{x}$:

$$\widetilde{X}(f) = \Psi_x(f) Y(f) \qquad (\Psi_x \in \mathbb{C}) \quad (4)$$

$$\text{minimize } E = \int \underbrace{\left|\widetilde{X} - \widehat{X}\right|^2}_{=I} df \quad (5)$$

$$\rightarrow \frac{\partial E}{\partial \Psi_x} = 0 \quad (6)$$

$$I = \left|\widetilde{X} - \widehat{X}\right|^2 = \left|\Psi_x Y - \frac{Y-N}{H}\right|^2$$

$$= \left|\left(\Psi_x - \frac{1}{H}\right) Y + \frac{N}{H}\right|^2$$

$$= \left|\left(\Psi_x - \frac{1}{H}\right) Y\right|^2 + \frac{|N|^2}{|H|^2}$$

$$+ \left(\Psi_x - \frac{1}{H}\right) Y \frac{N^*}{H^*} + \left(\Psi_x^* - \frac{1}{H^*}\right) Y^* \frac{N}{H}$$

$$\left(\begin{array}{l} \left(\Psi_x - \frac{1}{H}\right) Y \frac{N^*}{H^*} \qquad \text{(Eq.(2))} \\ = \left(\Psi_x - \frac{1}{H}\right) (H\widehat{X} + N) \frac{N^*}{H^*} \\ = \underbrace{\left(\Psi_x - \frac{1}{H}\right) \frac{H}{H^*} \widehat{X} N^*}_{\text{Integral}=0} + \left(\Psi_x - \frac{1}{H}\right) \frac{1}{H^*} |N|^2 \end{array}\right)$$

$$E = \int I(f)\, df = \int I'(f)\, df$$

$$I' = \left|\left(\Psi_x - \frac{1}{H}\right) Y\right|^2 - \frac{|N|^2}{|H|^2}$$

$$+ \Psi_x \frac{|N|^2}{H^*} + \Psi_x^* \frac{|N|^2}{H}$$

Wiener deconvolution

$$\frac{\partial E}{\partial \Psi_x} = 0 \text{ or } \frac{\partial E}{\partial \Psi_x^*} = 0$$

$$E = \int I'(f)\, df$$

$$I' = \left| \left(\Psi_x - \tfrac{1}{H}\right) Y \right|^2 - \tfrac{|N|^2}{|H|^2} + \Psi_x \tfrac{|N|^2}{H^*} + \Psi_x^* \tfrac{|N|^2}{H}$$

$$\left(\begin{array}{l} \frac{\partial}{\partial \Psi_x} \left(\left| \left(\Psi_x - \frac{1}{H}\right) Y \right|^2 \right) \\ \quad = \frac{\partial}{\partial \Psi_x} \left((\Psi_x^* - \frac{1}{H^*})(\Psi_x - \frac{1}{H}) |Y|^2 \right) \\ \quad = \left(\Psi_x^* - \frac{1}{H^*}\right) |Y|^2 \end{array} \right)$$

$$\frac{\partial I'}{\partial \Psi_x^*} = \left(\Psi_x - \tfrac{1}{H}\right) |Y|^2 + \tfrac{|N|^2}{H} = 0$$

$$\rightarrow \quad \Psi_x = \frac{1}{H} \underbrace{\frac{|Y|^2 - |N|^2}{|Y|^2}}_{=\Phi_y \text{ if } |N|=|N'|.}$$

Wiener deconvolution :

$$\Psi_x(f) = \frac{\Phi_y(f)}{H(f)} \quad (7)$$

$$\Phi_y(f) = \frac{|Y|^2 - |N'(f)|^2}{|Y(f)|^2} \quad (8)$$

$$\widetilde{X}(f) = \Psi_x(f) Y(f) = \frac{1}{H(f)} \Phi_y(f) Y(f) \quad (9)$$

Spectrum of Wiener deconvolution is equivalent to the divided spectrum of the spectrum applied Wiener filter to $Y(f)$.

In the ideal case of $N'(f) = N(f)$

$$
\begin{aligned}
&(N' = N)\\
&\Psi_x = \frac{1}{H}\Phi_y\\
&\quad = \frac{1}{H}\frac{|Y|^2-|N|^2}{|Y|^2} = \frac{1}{H}\frac{\left|\widehat{Y}\right|^2}{\left|\widehat{Y}\right|^2+|N|^2}\\
&\quad = \frac{H^*\left|\widehat{X}\right|^2}{|H|^2\left|\widehat{X}\right|^2+|N|^2}
\end{aligned}
\tag{10}
$$

- $H \neq 0$ and $N = 0$:

  $$\Psi_x = \frac{1}{H} \quad \rightarrow \quad \widetilde{X} = \frac{Y}{H} = \widehat{X}$$

  (Ideal case)

- $H = 0$ and $N \neq 0$:

  $$\Psi_x = 0 \quad \rightarrow \quad \widetilde{X} = 0$$

  (Cannot restore)

- $H = 0$ and $N = 0$:

  $$\lim_{H\to 0}\left\{\lim_{N\to 0}\left|\frac{H^*\left|\widehat{X}\right|^2}{|H|^2\left|\widehat{X}\right|^2+|N|^2}\right|\right\} = \lim_{H\to 0}\left\{\left|\frac{1}{H}\right|\right\} = \infty$$

  $$\lim_{N\to 0}\left\{\lim_{H\to 0}\left|\frac{H^*\left|\widehat{X}\right|^2}{|H|^2\left|\widehat{X}\right|^2+|N|^2}\right|\right\} = 0$$

  Estimation is impossible.
  (It takes different values, if the path to take limitation is different.)

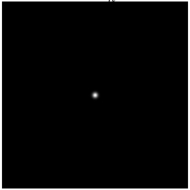
If $H = 0$, $\widetilde{X}$ cannot be detemined.
$\rightarrow$ (Consider as $\widetilde{X}(f) = 0$ to obtain $\widetilde{x}(t)$.)

Example of Wiener deconvolution (Input : $\sigma_h = 5, \sigma_n = 5$)

$\widehat{x}(\boldsymbol{r})$

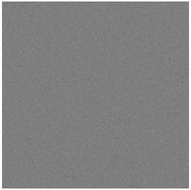
$[0, 255]$

$h(\boldsymbol{r}) = \frac{1}{2\pi\sigma_h^2} e^{-\frac{r^2}{2\sigma_h^2}}$



$[0, 1/(2\pi\sigma_h^2)]$

$n(\boldsymbol{r})$

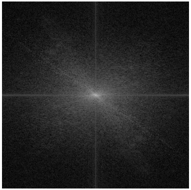


$[-64, 64]$

$y(\boldsymbol{r}) = h * \widehat{x} + n$

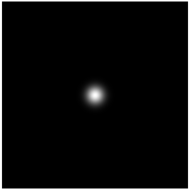
$[0, 255]$

$\widehat{X}(\boldsymbol{k})$



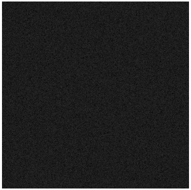
$\log\ [1, 10^4]$

$H(\boldsymbol{k}) = e^{-\frac{\sigma_h^2 k^2}{2}}$



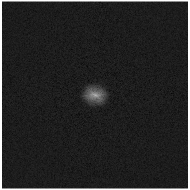
$[0, 1]$

$|N(\boldsymbol{k})|$



$\log\ [1, 10^4]$

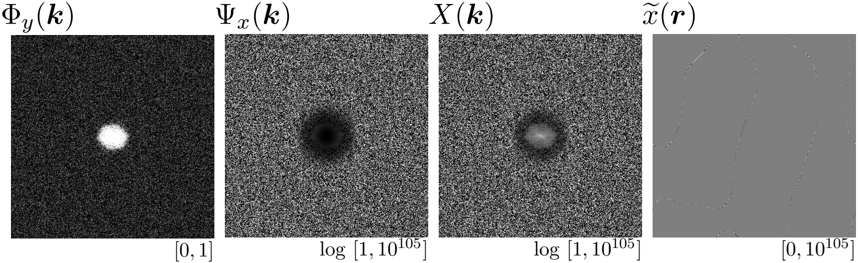
$Y(\boldsymbol{k}) = H\, \widehat{X} + N$



$\log\ [1, 10^4]$

Example of Wiener deconvolution (Result (failed case))

$(|N'(f)| = \sigma'_n = 5)$



Since $|\Psi_x(\boldsymbol{k})|$ includes very large spectrum, $\widetilde{X}(\boldsymbol{k})$ diverges, and $\widetilde{x}(\boldsymbol{r})$ diverges.

Reason of large filter gain.

- $H = 0$ and $N' \neq N$

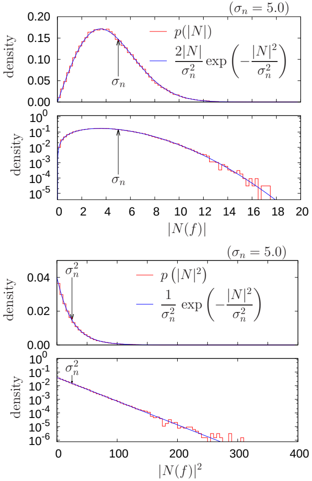
$$
\begin{aligned}
\Psi_x &= \lim_{H\to 0} \frac{1}{H}\Phi_y \\
&= \lim_{H\to 0} \frac{1}{H} \frac{\mathrm{Pos}\left\{|Y|^2 - |N'|^2\right\}}{|Y|^2} \\
&= \lim_{H\to 0} \frac{1}{H} \frac{\mathrm{Pos}\left\{\left|H\widehat{X}+N\right|^2 - |N'|^2\right\}}{\left|H\widehat{X}+N\right|^2} \\
&= \lim_{H\to 0} \frac{1}{H} \frac{\mathrm{Pos}\left\{|N|^2 - |N'|^2\right\}}{|N|^2}
\end{aligned}
$$

$$\mathrm{Pos}\,\{F\} \equiv \max\{F, 0\}$$

$$|\Psi_x| = \begin{cases} \infty & (|N| > |N'|) \\ 0 & (|N| \leq |N'|) \end{cases}$$

$\rightarrow$ Not continuous

Prob. dens. of $N(f)$



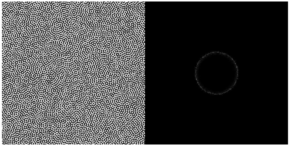
Methods to supress divergence of filter gain.

- Specify large $|N'|$
  $$\Psi'_x = \frac{1}{H}\frac{\mathrm{Pos}\left\{|Y|^2 - \alpha|N'|^2\right\}}{|Y|^2} \quad (\alpha > 1)$$
- Limitation of domain
  $$\Psi'_x(\boldsymbol{k}) = \Theta\left\{|k| \le k_{\max}\right\}\Psi_x(\boldsymbol{k})$$
  $$\left(\Theta\{\mathrm{C}\} \equiv \begin{cases} 1 & (\text{C is true}) \\ 0 & (\text{C is false}) \end{cases}\right)$$
- Limitation of range
  $$\Psi'_x = \Theta\left\{|\Psi(k)| \le \Psi_{\max}\right\}\Psi_x(\boldsymbol{k})$$
- Limitation of $|H|$
  $$\Psi'_x = \Theta\left\{|H| > H_{\min}\right\}\Psi_x$$
- Inspection of neighbores
  $$\Psi'_x(\boldsymbol{k}) = \Theta\left\{\frac{M_0(\boldsymbol{k})}{M_a(\boldsymbol{k})} > r_{0_{\min}}\right\}\Psi_x(\boldsymbol{k})$$
  - $M_a(\boldsymbol{k})$ is the number of pixels negiboring $\boldsymbol{k}$.
  - $M_0(\boldsymbol{k})$ is the number of pixels with $\Phi_y(\boldsymbol{k}') = 0$ within the pixels negiboring $\boldsymbol{k}$.

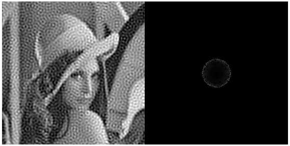
Suppression of divergence in Wiener Deconvlution (e.g.)

Before Deconv. True

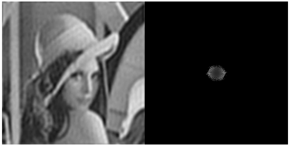
$k_{\max} = 0.15$: Divergent



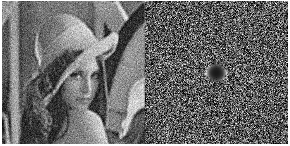
$k_{\max} = 0.10$: Periodic Pattern



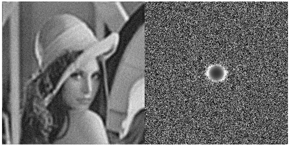
$\alpha = 4$: Blurring



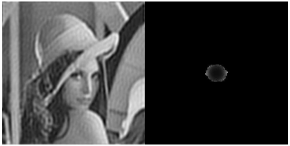
$\Psi_{\max} = 10$: Noisy



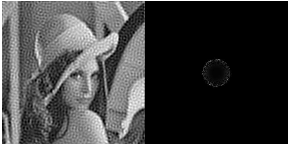
$\Psi_{\max} = 5$: Noisy



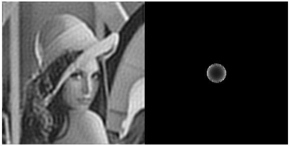
Insp. Neigh. : Ringing



$H_{\min} = 0.01$: Peri. Patt.



$H_{\min} = 0.1$: Peri. Patt., Ring.



Reason of errors in Wiener deconvolution

- Divergent

  Insufficient reduction of filter gain for small H.

- Blurring

  Over filtering of high freq. component.

- Periodic Pattern

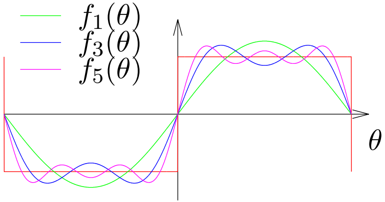
  Insufficient filtering for a certain component.

- Noisy

  Insufficient filtering of high freq. component.

- Ringing

  Ghost appeares along edges.
  Caused by Fourier transform wiht finite terms' trancation.
  $\Leftrightarrow$ Gibbs phenomenon
  (Impossible to avoid ringing.)



8.3 Estimation of Response function $H(f)$

$$Y(f) = H(f) \cdot \widehat{X}(f) + N(f)$$

$$\widetilde{X}(f) = \frac{1}{H(f)} \Phi_y(f) Y(f)$$

$H(f)$: Function is known, but parameter of the function is unknown.

$$\left(\text{e.g.: } H(f) = e^{-\frac{f^2}{2\sigma_f^2}} \quad (\sigma_f \text{ is unknown}) \right)$$

- Determine the parameter by least square fitting, and compute $\widetilde{H}(f)$
- The data for the least square fitting is flatten data by noise probability density function in the domain where the noise is dominant.

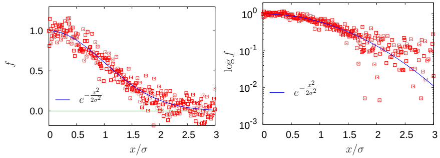
$$\widetilde{X}'(f) = \frac{1}{\widetilde{H}(f)} \Phi_y(f) Y(f)$$

Least square fitting to Gaussian function

- Fitting function (Estimated Value)
  $$\widetilde{f}(x_i;\, a, b) = ae^{-bx_i^2}$$
- Observed value
  $$(x_i, f_i) \quad i \in \{1, \cdots, N\}$$
- Residual $\quad \Delta f_i = f_i - \widetilde{f}(x_i)$
- Average of Square Residual
  $$E = \overline{(\Delta f)^2}$$
- minimize E

- In general method, the normal equations become non-linear equations.
  $\rightarrow$Complex

- Mapping from f to $F = \log f$.
  $$\widetilde{F}_i(x_i; a, b) = \log a - bx_i^2$$
  - quadratic equation
  - Expacted error for f and that for F are different.
    $\rightarrow$ It should be consider the weight.
    $$E = \overline{f^2(\Delta \log f)^2}$$



Weighted least square method

Weight $\equiv$ reliability of data

$$E = \overline{w_i \left(f_i - \widetilde{f_i} \right)^2}$$

Low reliability of the data with the large scattering.

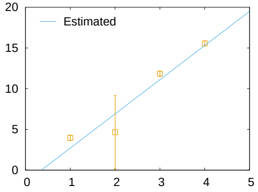
$$\mathrm{E}\left[\left(f_i - \widetilde{f_i} \right)^2 \right] \sim \mathrm{E}\left[\left(f_i - \widehat{f_i} \right)^2 \right] = \sigma_{f_i}^2$$

$$\to \frac{1}{\sigma_{f_i}^2} \mathrm{E}\left[\left(f_i - \widetilde{f_i} \right)^2 \right] \sim 1(\text{constant})$$

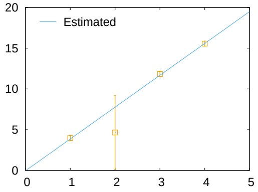
$$\to \text{Independent of } i \quad \to \quad \therefore w_i = \frac{1}{\sigma_{f_i}^2}$$

$$E = \overline{\frac{1}{\sigma_{f_i}^2} \left(f_i - \widetilde{f_i} \right)^2} \qquad (1)$$

- Without weight



- With weight



Mapping of variable in least square method

$$E = \overline{\frac{1}{\sigma_{f_i}^2}\left(f_i - \widetilde{f_i}\right)^2} \qquad (2)$$

In the case of $F_i = F(f_i)$,

$$f_i - \widetilde{f_i} = \Delta f_i = \Delta F_i \frac{\Delta f_i}{\Delta F_i}$$

$$\simeq (F_i - \widetilde{F_i}) \left.\frac{df}{dF}\right|_i$$

$$E = \overline{\frac{1}{\left(\frac{dF}{df}\right)_i^2 \sigma_{f_i}^2}\left(F_i - \widetilde{F_i}\right)^2} \qquad (3)$$

$\left|\frac{dF}{df}\right|$ shows a magnification factor of error bar.

If $F = \log f$,

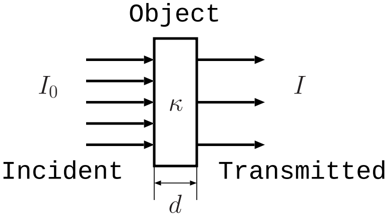
$$\frac{dF}{df} = \frac{1}{f}$$

$$E = \overline{\frac{f_i^2}{\sigma_{f_i}^2}(\log f_i - \log \widetilde{f_i})^2}$$

- $f = \widehat{f} + n$
  n: white $\rightarrow \sigma_n^2 = \mathrm{const.}$
- $E = \frac{1}{\sigma_n^2}\overline{f_i^2 (\log f_i - \log \widetilde{f_i})^2}$
- Larger f has larger weight.

9. Computed Tomography (CT)
9.1 Absorption of X-ray

- X-ray radiography



$$I(y) = I_0 e^{-\kappa(y)d}$$

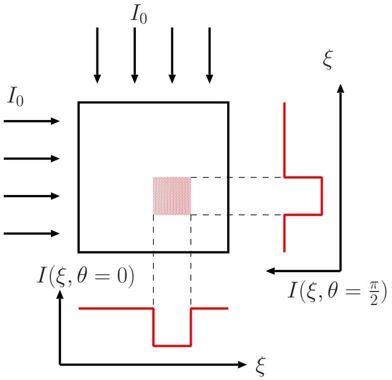
- κ : attenuation coefficient
  In the case of X-ray,
  it depends on the atomic number.
  (Heavy atom $\rightarrow$ large κ.)

- If κ is depth dependent,

$$\kappa d \rightarrow \int_0^d \kappa(x,y)\,dx$$
$$\rightarrow \int_{-\infty}^{\infty} \kappa(x,y)\,dx$$
$$(\kappa(x,y) = 0 \quad \text{Not Object})$$

- Only the integral of κ along optical path can be obtained from X-ray radiography.

9.2 Projection from several directions

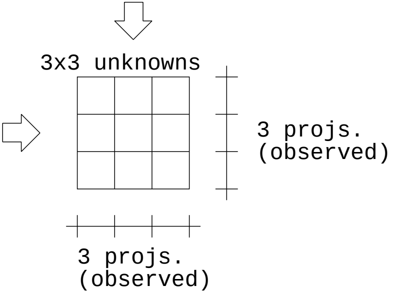


The distribution of κ including depth distribution, which is called tomography, can be obtained from several projected data with different directions.

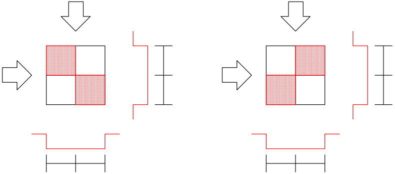
$$\Downarrow$$

Computed Tomography

Number of projections and Num. of internal nodes



Num. of unknowns > Num. of Obs.
→ Cannot solve.



Num. of unknowns = Num. of Obs.
→ Cannot distinguish.

To obtain more projection, other projections with different directions are needed.

$$\theta \in [0, \pi]$$

9.3 Schematic of forward and backward-projection
Forward projection (measuring process)

Forward projection

$\equiv$Integral along beam path

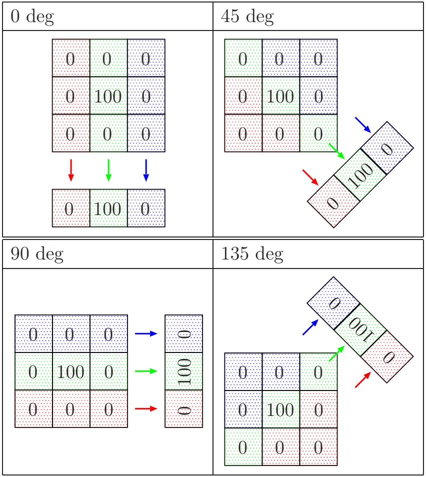
$$p(\xi,\theta) = \int_\eta \kappa(x,y)\, d\eta_\theta$$

$$(x \equiv x(\xi,\theta),\ \ y \equiv y(\xi,\theta))$$

$\rightarrow$ Accumulate along path

0	0	0
0	100	0
0	0	0

$\Longrightarrow$



Backward projection (1) Simple backprojection

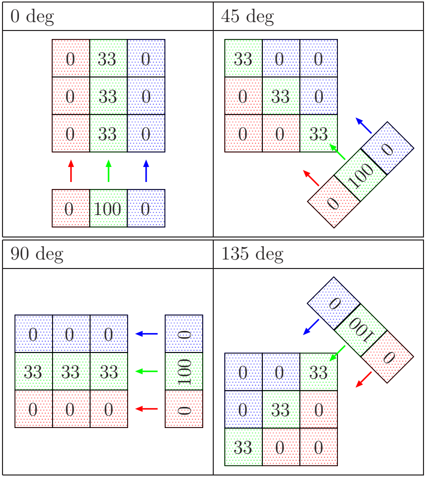
1. Map averaged value of the projection data along path for each angle. ⟹
2. Take an average of mapped data for each pixel. ⟸

8	8	8
8	33	8
8	8	8

Blurrier than original.

Original

0	0	0
0	100	0
0	0	0



Backward projection (2) Filtered backward projection

In the simple backward projection, the reconstracted result is blurred. To reduce the blurring, edge enhancement filter is applied to the projection data.

Edge enhancement filter

$$\begin{aligned} g_n &= f_n - k f_n'' \\ &= f_n - k(f_{n-1} - 2f_n + f_{n+1}) \\ &= -k f_{n-1} + (2k+1) f_n - k f_{n+1} \end{aligned}$$

$(k = 1)$

$$g_n = \sum_{i=-1}^{+1} w_i f_{n-i}$$

$$(w_{-1}, w_0, w_{+1}) = (-1, +3, -1)$$

Filtered backward projection

1. Edge enhancement of projection data: p'

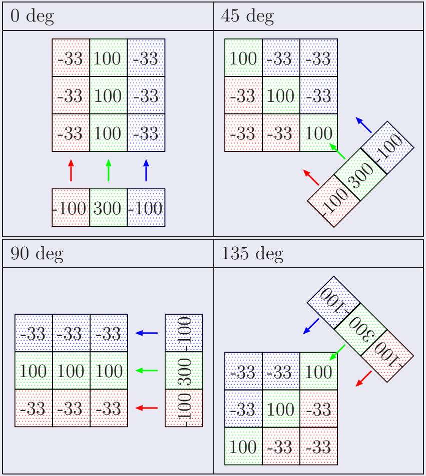
   (e.g.) $p'_j = \sum_{i=-1}^{+1} w_i p_{j-i}$

   $(w_{-1}, w_0, w_{+1}) = (-1, 3, -1)$

p	0	100	0
p'	-100	300	-100

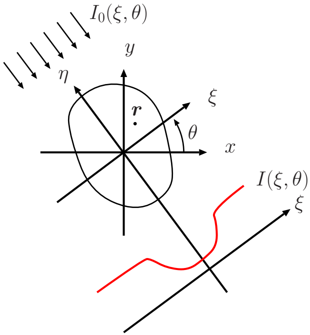
2. Apply simple backward projection using p'.

0	0	0
0	100	0
0	0	0



In this example, the reconstructed field is identical to original.

9.4 Radon Transform
Coordinate system



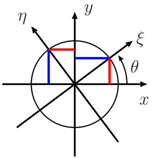
Expression of point $\boldsymbol{r}$

$$\boldsymbol{r} = x\,\boldsymbol{e}_x + y\,\boldsymbol{e}_y$$
$$= \xi\,\boldsymbol{e}_{\boldsymbol{\xi}} + \eta\,\boldsymbol{e}_{\boldsymbol{\eta}}$$

Inner product of $\boldsymbol{e}_x$:

$$x\,\boldsymbol{e}_x\cdot\boldsymbol{e}_x + y\,\boldsymbol{e}_x\cdot\boldsymbol{e}_y$$
$$= \xi\,\boldsymbol{e}_x\cdot\boldsymbol{e}_{\boldsymbol{\xi}} + \eta\,\boldsymbol{e}_x\cdot\boldsymbol{e}_{\boldsymbol{\eta}}$$

$$x = \xi\cos\theta - \eta\sin\theta$$



$$\begin{cases} x = +\xi\cos\theta - \eta\sin\theta \\ y = +\xi\sin\theta + \eta\cos\theta \end{cases}$$

$$\begin{cases} \xi = +x\cos\theta + y\sin\theta \\ \eta = -x\sin\theta + y\cos\theta \end{cases}$$

Radon Transform

- Projected data (known)

$$I(\xi,\theta) = I_0\, e^{-\int_L \kappa(\boldsymbol{r})\,dl}$$
$$L \in \{\boldsymbol{r}(\xi,\eta;\theta)\,|\,\xi = \xi'(\text{const.})\}$$

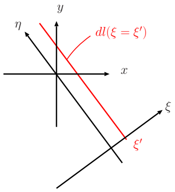
- Sinogram (known)

$$p(\xi,\theta) \equiv -\log\frac{I(\xi,\theta)}{I_0}$$

- Radon Transform
  Integral over straight line in 2-D space.

$$p(\xi,\theta) = \int_L \kappa(\boldsymbol{r}(\xi,\theta))\,dl$$

- Extend from line integral to 2-D area integral
  $\xi' = x\cos\theta + y\sin\theta$



$$\int_L [\cdots]\,dl = \int_{\xi=\xi'} [\cdots]\,d\eta$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\cdots]\delta(\xi-\xi')\,d\xi'\,d\eta$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\cdots]\delta(\xi-\xi'(x,y))\,dx\,dy$$
$$= \iint[\cdots]\delta(\xi-(x\cos\theta+y\sin\theta))\,dx\,dy$$

9.5 Projection slice theorem

- Projection (Sinogram)

$$p(\xi,\theta) = \iint \kappa(x,y)\delta(\xi - (x\cos\theta + y\sin\theta))\,dx\,dy \qquad (1)$$

- FT with respect to ξ

$$\begin{aligned} P(k_\xi,\theta) &= \iiint \kappa(x,y)\delta(\xi - (x\cos\theta + y\sin\theta))e^{-jk_\xi \xi}\,dx\,dy\,d\xi \\ &= \iint \kappa(x,y)e^{-jk_\xi(x\cos\theta + y\sin\theta)}\,dx\,dy \qquad (2) \end{aligned}$$

2-D Fourier Transform in polar coordinate system

- Forward transform

$$F(k_x, k_y) = \iint f(x,y) e^{-j(k_x x + k_y y)}\, dx\, dy \qquad (k_x = k\cos\theta, \quad k_y = k\sin\theta)$$

$$= \iint f(x,y) e^{-jk(x\cos\theta + y\sin\theta)}\, dx\, dy \equiv F'(k,\theta) \tag{3}$$

- Inverse transform

$$f(x,y) = \frac{1}{4\pi^2} \iint F(k_x, k_y) e^{+j(k_x x + k_y y)}\, dk_x\, dk_y \qquad \left(\iint dk_x\, dk_y = \int_0^\infty \int_0^{2\pi} k\, d\theta\, dk\right)$$

$$= \frac{1}{4\pi^2} \int_0^\infty \int_0^{2\pi} F'(k,\theta) e^{+jk(x\cos\theta + y\sin\theta)}\, k\ d\theta\, dk \tag{4}$$

Eq. (2) and Eq. (3) are same.

Projection Slice Theorem

$P(k_\xi, \theta)$ is expressed by Fourier transform of $\kappa(x, y)$ in polar coordinate system.

Since $P(k_\xi, \theta)$ is known, $\kappa(x, y)$ is obtained by the inverse FT using (4).

$$\kappa(x, y) = \frac{1}{4\pi^2} \int_0^\infty \int_0^{2\pi} P(k_\xi, \theta) e^{+jk_\xi(x\cos\theta + y\sin\theta)}\, k_\xi \; d\theta\, dk_\xi \qquad (5)$$

Projection from opposite direction

$$\kappa(x,y) = \frac{1}{4\pi^2}\int_0^\infty\int_0^{2\pi} P(k_\xi,\theta)e^{+jk_\xi(x\cos\theta+y\sin\theta)}\, k_\xi \; d\theta\, dk_\xi$$
$$= \frac{1}{4\pi^2}\int_{-\infty}^\infty\int_0^{\pi} P(k_\xi,\theta)e^{+jk_\xi(x\cos\theta+y\sin\theta)}\, |k_\xi| \; d\theta\, dk_\xi$$

$$\int_0^\infty\int_0^{2\pi} k_\xi\, d\theta\, dk_\xi = \int_0^\infty\int_0^{\pi} k_\xi\, d\theta\, dk_\xi + \int_0^\infty\int_\pi^{2\pi} k_\xi\, d\theta\, dk_\xi$$

Projection from opposite direction

$$\begin{pmatrix} \theta \to \theta \pm \pi \\ \xi \to -\xi \end{pmatrix} \quad \begin{pmatrix} p(\xi,\theta\pm\pi) = p(-\xi,\theta) \\ P(k_\xi,\theta\pm\pi) = P(-k_\xi,\theta) \\ e^{+jk_\xi(x\cos(\theta\pm\pi)+y\sin(\theta\pm\pi))} = e^{-jk_\xi(x\cos\theta+y\sin\theta)} \end{pmatrix}$$

2nd Term $= \int_0^\infty\int_\pi^{2\pi} P(k_\xi,\theta)e^{+jk_\xi(x\cos\theta+y\sin\theta)}k_\xi\, d\theta\, dk_\xi \qquad (\theta' = \theta - \pi)$

$$= \int_0^\infty\int_0^{\pi} P(-k_\xi,\theta')e^{-jk_\xi(x\cos\theta'+y\sin\theta')}k_\xi\, d\theta'\, dk_\xi \qquad (k'_\xi = -k_\xi)$$
$$= \int_0^{-\infty}\int_0^{\pi} P(k'_\xi,\theta')e^{+jk'_\xi(x\cos\theta'+y\sin\theta')}(-k'_\xi)\, d\theta'\, (-dk'_\xi)$$
$$= \int_{-\infty}^{0}\int_0^{\pi} P(k'_\xi,\theta')e^{+jk'_\xi(x\cos\theta'+y\sin\theta')}|k'_\xi|\, d\theta'\, dk'_\xi$$

9.6 Reconstruction by using Fourier transform

$$\kappa(x,y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{0}^{\pi} P(k_\xi,\theta)e^{+jk_\xi \overbrace{(x\cos\theta + y\sin\theta)}^{\xi}}|k_\xi|\; d\theta\, dk_\xi$$

$$= \frac{1}{2\pi}\int_0^{\pi} \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty}[P(k_\xi,\theta)|k_\xi|]\, e^{+jk_\xi \xi}\, dk_\xi}_{=\mathcal{F}_{k_\xi}^{-1}\left\{P((k_\xi,\theta)|k_\xi|\right\}\equiv q(\xi,\theta)}\, d\theta = \frac{1}{2}\underbrace{\frac{1}{\pi}\int_0^{\pi} q(\xi(x,y),\theta)\, d\theta}_{\text{Average with } \theta.}$$

$$q(\xi(x,y),\theta) = \int_{-\infty}^{\infty}[P(k_\xi,\theta)|k_\xi|]\, e^{+jk_\xi(x\cos\theta + y\sin\theta)}\, dk_\xi$$

$$\kappa(x,y) = \frac{1}{2}\left\langle \mathcal{F}_{k_\xi}^{-1}\left\{\mathcal{F}_\xi\left\{p(\xi,\theta)\right\}_\xi H(k_\xi)\right\}_{k_\xi}\right\rangle_\theta$$

$(H(k_\xi) = |k_\xi|$ in the case of Ramp function)

1. Sinogram :

   $$p(\xi, \theta)$$

2. FWD FT with ξ :

   $$P(k_\xi, \theta) = \int_{-\infty}^{\infty} p(\xi, \theta) e^{-jk_\xi \xi} \, d\xi$$

3. Filtering (Weight with $|k_\xi|$) :

   $$|k_\xi| P(k_\xi, \theta)$$

4. INV FT with k_ξ :

   $$q(\xi, \theta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} P(k_\xi, \theta) |k_\xi| e^{+jk_\xi \xi} \, dk_\xi$$

5. Backward projection (Coordinate transform and integrate with θ) :

   $$\kappa(x, y) = \frac{1}{2\pi} \int_0^{\pi} q(x \cos\theta + y \sin\theta, \theta) \, d\theta$$

Two FT (FWD and INV) are needed for a certain θ.

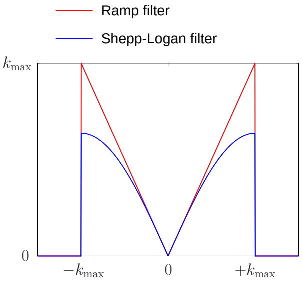
Filtered Back-Projection

- Before IFT, $|k_\xi|$ is multiplied.
- Since this factor $|k_\xi|$ is considered as a filter in the spectral domain, the method based on FT is called
  Filtered Back-projection (FBP).
- In the actual computation, $k_\xi \in [-\infty, \infty] \to [-k_{\max}, +k_{\max}]$.
- $k_{\max}$ is Nyquist frequency determined by sampling interval.

- To avoid ringing artifact caused by high frequency component, another filter can be applied.

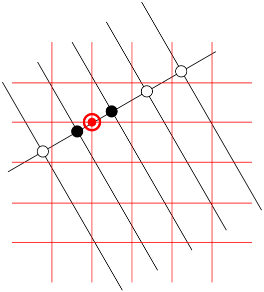
(e.g. Shepp-Logan filter)

$$H(k_\xi) = \frac{2k_{\max}}{\pi} \sin\left|\frac{\pi k_\xi}{2k_{\max}}\right|$$



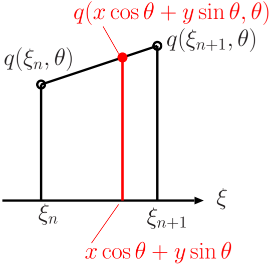
Handling of discrete data

- $p(\xi_n, \theta_m) = p(n\Delta\xi, m\Delta\theta)$
  (n, m) : Integer
- $\kappa(x_i, y_j) = \kappa(i\Delta x, j\Delta y)$ (i, j) : Integer



- In order to evaluate $q(x\cos\theta + y\sin\theta, \theta)$ from $q(\xi, \theta)$ interpolation are needed.

e.g. (Interpolation for ξ)



Reconstruction using convolution

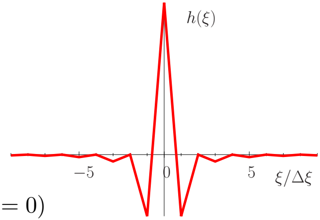
$$\left\{ \begin{array}{l} c(x) = \int a(x')b(x-x')\,dx' \\ C(k) = A(k)B(k) \end{array} \right\} \Leftrightarrow \quad c(x) = \frac{1}{2\pi}\int A(k)B(k)e^{+jkx}\,dk$$

$$q(\xi,\theta) = \frac{1}{2\pi}\int_{-k_{\max}}^{k_{\max}} P(k_\xi,\theta)H(k_\xi)e^{+jk_\xi\xi}\,dk_\xi = \int_{-\xi_{\max}}^{\xi_{\max}} p(\xi',\theta)h(\xi-\xi')\,d\xi'$$

- If $H(k_\xi) = |k_\xi|$,

$$h(\xi) = \frac{1}{2\pi}\int_{-k_{\max}}^{k_{\max}} |k_\xi| e^{+jk_\xi\xi}\,dk_\xi$$

$$= \begin{cases} \frac{1}{2\pi}k_{\max}^2 & (\xi = 0) \\ \frac{1}{\pi}\frac{k_{\max}}{\xi}\sin(k_{\max}\xi) + \frac{1}{\xi^2}(\cos(k_{\max}\xi) - 1) & (\xi \neq 0) \end{cases}$$



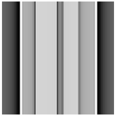
Subtract by neighbors ⇔ Edge Enhancement

1. Sinogram : $p(\xi, \theta)$
2. Convolution :
$$q(\xi, \theta) = \int p(\xi', \theta) h(\xi - \xi') \, d\xi'$$
3. Back-projection :
$$\kappa(x, y) = \frac{1}{2\pi} \int_0^{\pi} q(x \cos\theta + y \sin\theta, \theta) \, d\theta$$

$\kappa(x, y)$

$p'(x, y; 0)$

$q'(x, y; 0)$



$\left(\begin{array}{l} p', q' \text{ : contrast is enhanced to display} \\ (\mathrm{erf}(I/\sigma)) \end{array} \right)$

Only one convolution for each θ.
No FT.

Number of multiplications for each θ		
Fourier transform	DFT × 2	$2N^2$
	FFT × 2	$2N \log N$
Convolutinal integral	All points	N^2
	Neighboring M pts. $(M \ll N)$	MN

$\rightarrow$ Faster computation than DFT, if the convolution is applied to neighboring points only.

Filtered Back-projection and Simple BP

- Filtered Back-Projection

$$\kappa(x,y) = \frac{1}{2}\left\langle \mathcal{F}^{-1}_{k_\xi}\left\{\mathcal{F}_\xi\left\{p(\xi,\theta)\right\}_\xi H(k_\xi)\right\}_{k_\xi}\right\rangle_\theta$$

- Simple Back-projection

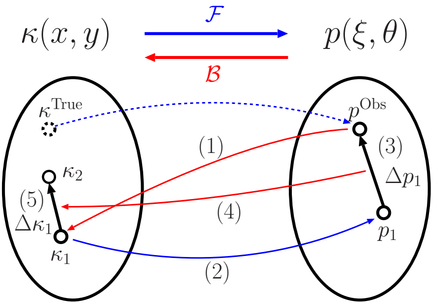
$$\begin{aligned} H(k) &= 1 \\ \kappa(x,y) &= \frac{1}{2}\left\langle \mathcal{F}^{-1}_{k_\xi}\left\{\mathcal{F}_\xi\left\{p(\xi,\theta)\right\}_\xi\right\}_{k_\xi}\right\rangle_\theta = \frac{1}{2}\langle p(\xi,\theta)\rangle_\theta \\ &= \frac{1}{2\pi}\int_0^\pi p(\xi,\theta)\,d\theta \end{aligned}$$

  - No need to FT$\rightarrow$ Fast
  - The reconstructed distribution is blurred.

$\rightarrow$ Iterate two procedures of projection and back-projection.

9.7 Iterative Reconstruction

Applying the forward projection (FP) to the reconstructed field obtained by the backward projection (BP), we can evaluate the error.
The BP of the error is added to the field obtained in the previous step. The simple BP is used for the BP algorithm, since the simple BP is fast.



(1) BP for the proj. : $\kappa_1 = \mathcal{B}\left\{p^{\text{Obs}}\right\}$

(2) FP for the field : $p_1 = \mathcal{F}\left\{\kappa_1\right\}$

(3) Under-estimation : $\Delta p_1 = p^{\text{Obs}} - p_1$

(4) BP for the under-est. : $\Delta\kappa_1 = \mathcal{B}\left\{\Delta p_1\right\}$

(5) Update the field : $\kappa_2 = \kappa_1 + \alpha\Delta\kappa_1$

(α : Relaxation factor for stable reconstruction $0 < \alpha \leq 1$)

Simple Back-projection

- Sinogram

$$p(\xi_n,\theta_m) = \int_{L_{nm}} \kappa(x,y)\,dl \simeq \overline{\kappa(x,y\in L_{nm})}\Delta l_{nm}$$

$$\to \overline{\kappa(x,y\in L_{nm})} = \frac{p(\xi_n,\theta_m)}{\Delta l_{nm}} \qquad \left(\Delta l_{nm} = \int_{L_{nm}} dl\right)$$

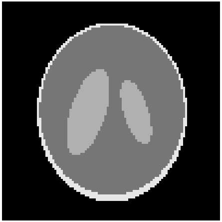
- Fraction of projection is mapped onto the internal distribution.

$$\kappa(x_i,y_j) = \frac{1}{N_n}\sum_m \sum_n a_{i,j,n,m}\frac{p(\xi_n,\theta_m)}{\Delta l_{nm}}$$

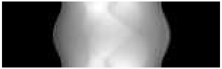
$a_{i,j,n,m}$: Overlap area fraction between the pixel (x_i,y_j) and the beam L_{nm} with width $\Delta\xi$.

9.8 Example of reconstruction by simulation

True $\kappa(x,y)$



Sinogram $p(\xi,\theta)$

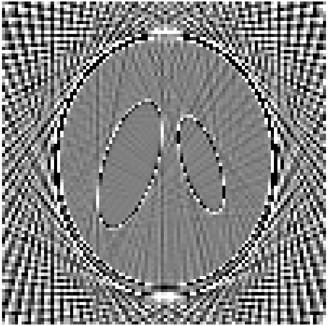


$\kappa(x,y) \in [0,2]$,
$N_x = N_y = 100$,
$N_\xi = 100$,
$N_\theta = 45(\Delta\theta = 4\text{deg})$

Filtered Back-Projection (Filter : Ramp function)

$\kappa(x,y)$ [0,2]

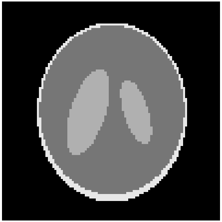
$\Delta\kappa(x,y)$ [-0.2,+0.2]



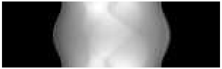
$$||\Delta\kappa||_2 \equiv \sqrt{\langle \Delta\kappa^2 \rangle} = 0.15$$

Line artifact

True $\kappa(x, y)$



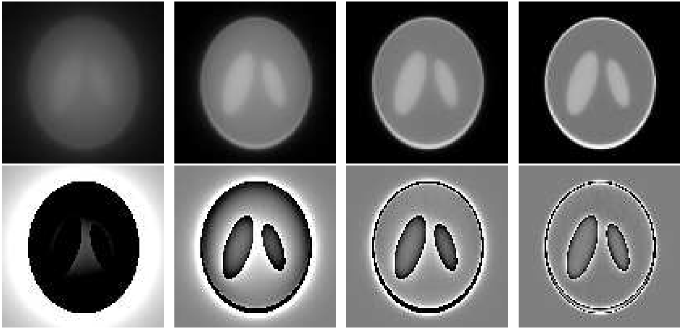
Sinogram $p(\xi, \theta)$



$\kappa(x, y) \in [0, 2]$,
$N_x = N_y = 100$,
$N_\xi = 100$,
$N_\theta = 45 (\Delta\theta = 4\text{deg})$

Iterative reconstruction

$N = 1$ | $N = 2$ | $N = 10$ | $N = 20$



$||\Delta\kappa||_2 = 0.42$ | $||\Delta\kappa||_2 = 0.25$ | $||\Delta\kappa||_2 = 0.17$ | $||\Delta\kappa||_2 = 0.11$

Reduction of edge blurring

10. Magnetic Resonance Imaging (MRI)

- Magnetic Resonance Imaging (MRI) :

  Measurement spatial distribution of specific atom using NMR.
  In medical application, the density of ^{1}H is measured.

  $$\text{MRI} \rightarrow \rho_{\mathrm{H}}(x, y, z)$$

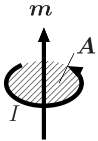
- NMR (Nuclear Magnetic Resonance) spectrometry:

  Analysis of properties of atoms or the molecules in a sample using NMR

  $$\text{NMR} \rightarrow s(|B|)$$

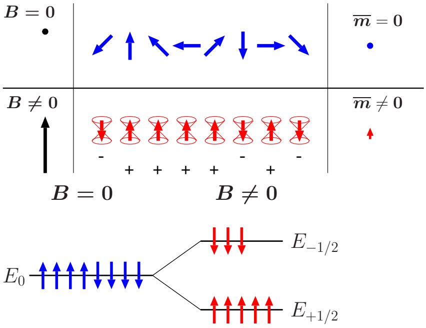
- Magnetic dipole moment $\boldsymbol{m}$

  $$\boldsymbol{m} = \mu_0 I \boldsymbol{A} = \gamma \hbar \boldsymbol{J} \qquad (1)$$



$$\left(\begin{array}{l} \gamma\text{:Gyromagnetic ratio; dependent on atom} \\ \hbar\text{:Planck constant} \\ \boldsymbol{J}\text{:Angular momentum } \hat{\boldsymbol{J}}^2 = J(J+1), \hat{J}_z = J \\ J\text{:Spin quantum number } (J = \frac{1}{2} \text{ for } ^{1}\text{H}) \end{array}\right)$$

10.1 Macroscopic Magnetic Dipole Moment



- $\boldsymbol{B} = \boldsymbol{0}$
  - Direction of individual magnetic moment $\boldsymbol{m}^{(i)}$ is random.
  - $\boldsymbol{M} = \overline{\boldsymbol{m}} = \boldsymbol{0}$

- $\boldsymbol{B} = B_0 \boldsymbol{e}_z \neq \boldsymbol{0}$
  - $\boldsymbol{m}(t)$ follows gyro motion.
  - $\boldsymbol{m}(t)_t$ is quantized by anomalous Zeeman effect.

$$E_{\pm\frac{1}{2}} = E_0 \mp \frac{1}{2}\hbar\gamma|\boldsymbol{B}|$$

  - Boltzmann distribution

$$n_{-\frac{1}{2}} = n_{+\frac{1}{2}} e^{\frac{-\Delta E}{kT}}$$
$$\to n_{+\frac{1}{2}} > n_{-\frac{1}{2}}$$
$$(\Delta n/n \sim 5\text{ppm})$$

  - $\boldsymbol{M} = \overline{\boldsymbol{m}} = |\overline{m}|\boldsymbol{e}_z \neq \boldsymbol{0}$

10.2 Precession of Magnetic Dipole

Motion of equation of magnetic dipole

(Derived from motion of equation due to Lorentz force ($\frac{d\boldsymbol{v}}{dt} \propto \boldsymbol{v} \times \boldsymbol{B}$, $\boldsymbol{v} \propto \boldsymbol{e}_\perp \times \boldsymbol{m}$))

$$\frac{d\boldsymbol{m}}{dt} = \gamma\, \boldsymbol{m} \times \boldsymbol{B} \qquad (2)$$

Case of $\boldsymbol{B} = B_0 \boldsymbol{e}_z \quad (|B_0| = \mathrm{const})$

From $\boldsymbol{e}_z \cdot$(Eq.(2))

$$\boldsymbol{e}_z \cdot \frac{d\boldsymbol{m}}{dt} = 0$$

$$m_z = \mathrm{const} \qquad (3)$$

From $\boldsymbol{m} \cdot$(Eq.(2))

$$\boldsymbol{m} \cdot \frac{d\boldsymbol{m}}{dt} = \frac{1}{2} \frac{d\boldsymbol{m} \cdot \boldsymbol{m}}{dt} = 0$$

$$\boldsymbol{m} \perp \frac{d\boldsymbol{m}}{dt} \qquad (4)$$

$$|\boldsymbol{m}|^2 = \mathrm{const} \qquad (5)$$

(Independent of $|\boldsymbol{B}|$)

From $\frac{d(\text{Eq.(2)})}{dt}$

$$\frac{d^2\boldsymbol{m}}{dt^2} = \gamma \frac{d\boldsymbol{m}}{dt} \times \boldsymbol{B} = (\gamma B_0)^2 (\boldsymbol{m} \times \boldsymbol{e}_z) \times \boldsymbol{e}_z = (\gamma B_0)^2 (-\boldsymbol{m} + m_z \boldsymbol{e}_z) \qquad (6)$$

From $\boldsymbol{e}_x\cdot$(Eq.(6)) and $\boldsymbol{e}_y\cdot$(Eq.(6))

$$\frac{d^2 m_n}{dt^2} + (\gamma B_0)^2 m_n = 0 \quad (n \in \{x, y\})$$

$$m_n = C_n \cos\omega_0 t + S_n \sin\omega_0 t \qquad (7)$$

$$\omega_0 = \gamma B_0 \qquad (8)$$

ω_0 : Larmor frequency

$\boldsymbol{e}_x\cdot$(Eq.(2)) : $\left(\frac{dm_x}{dt} = \gamma B_0 m_y\right)$
Substitute Eq.(7)

$$\omega_0(-C_x \sin\omega_0 t + S_x \cos\omega_0 t) = \underbrace{\gamma B_0}_{=\omega_0}(C_y \cos\omega_0 t + S_y \sin\omega_0 t)$$

$$\rightarrow -C_x = S_y, \quad S_x = C_y$$

Assume $\boldsymbol{m} = (m_\perp, 0)$ at $t = 0$, m_x.
From Eq.(7) at $t = 0$,

$$C_x = -S_y = m_\perp$$
$$C_y = S_x = 0$$

$\therefore$

$$m_x = +m_\perp \cos\omega_0 t \qquad (9)$$
$$m_y = -m_\perp \sin\omega_0 t \qquad (10)$$

$$m^2 = m_\perp^2 + m_z^2 = \text{const}$$

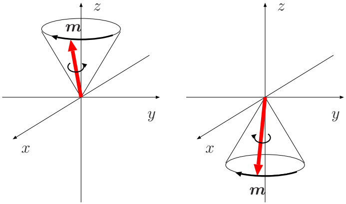
$\therefore$

$$m_z = \pm\sqrt{m^2 - m_\perp^2} \qquad (11)$$

- Precession

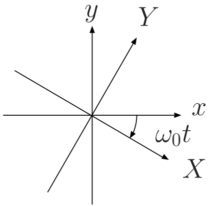
$$\frac{d\boldsymbol{m}}{dt} = \gamma\,\boldsymbol{m} \times (B_0 \boldsymbol{e}_z)$$

$$\begin{cases} m_x = +m_\perp \cos\omega_0 t \\ m_y = -m_\perp \sin\omega_0 t \\ m_z = \pm\sqrt{m^2 - m_\perp^2} \end{cases} \tag{12}$$



  - Circular motion around z-axis with clockwise direction

- Representation on Rotational coordinate



$$\begin{cases} \boldsymbol{e}_X(t) = \cos(\omega_0 t)\boldsymbol{e}_x - \sin(\omega_0 t)\boldsymbol{e}_y \\ \boldsymbol{e}_Y(t) = \sin(\omega_0 t)\boldsymbol{e}_x + \cos(\omega_0 t)\boldsymbol{e}_y \end{cases} \tag{13}$$

$$\boldsymbol{m}(t) = m_\perp \boldsymbol{e}_X(t) + m_z \boldsymbol{e}_z \tag{14}$$

10.3 Nuclear Magnetic Resonance(NMR)

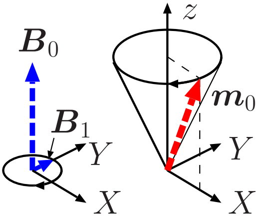
- Add a rotational magnetic field ($\omega = \omega_0$) with small amplitude,

$$\boldsymbol{B}(t) = \boldsymbol{B}_0 + \boldsymbol{B}_1(t) \tag{15}$$

$$\boldsymbol{B}_0 = B_0 \boldsymbol{e}_z$$

$$\boldsymbol{B}_1(t) = B_{1Y} \boldsymbol{e}_Y(t) \tag{16}$$

$$\left(\begin{array}{l} |\boldsymbol{B}_1(t)| \ll |\boldsymbol{B}_0| \\ B_{1Y} = \text{const}, \;\; B_{1X} = 0 \end{array} \right)$$



- Definition of $\boldsymbol{m}$.

$$\boldsymbol{m}(t) = m_X(t)\boldsymbol{e}_X(t) + m_Y(t)\boldsymbol{e}_Y(t) + m_z(t)\boldsymbol{e}_z \tag{17}$$

$$(\boldsymbol{m}(0) = m_\perp \boldsymbol{e}_X + m_{0z} \boldsymbol{e}_z)$$

- Time derivatives of unit vectors

$$\frac{d\boldsymbol{e}_X}{dt} = -\omega_0 \boldsymbol{e}_Y, \quad \frac{d\boldsymbol{e}_Y}{dt} = +\omega_0 \boldsymbol{e}_X \tag{18}$$

- Motion equation

$$\frac{d\boldsymbol{m}}{dt} = \gamma\, \boldsymbol{m} \times \boldsymbol{B}$$

  - l.h.s

$$\frac{d\boldsymbol{m}}{dt} = \begin{pmatrix} \omega_0 m_Y \\ -\omega_0 m_X \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{dm_X}{dt} \\ \frac{dm_Y}{dt} \\ \frac{dm_z}{dt} \end{pmatrix}$$

  - r.h.s

$$\gamma\, \boldsymbol{m} \times \boldsymbol{B} = \begin{vmatrix} m_X & m_Y & m_z \\ 0 & B_{1Y} & B_0 \\ \boldsymbol{e}_X & \boldsymbol{e}_Y & \boldsymbol{e}_z \end{vmatrix}$$

$$= \gamma \begin{pmatrix} m_Y B_0 \\ -m_X B_0 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} -m_z B_{1Y} \\ 0 \\ +m_X B_{1Y} \end{pmatrix}$$

$$\therefore \begin{pmatrix} \frac{dm_X}{dt} \\ \frac{dm_Y}{dt} \\ \frac{dm_z}{dt} \end{pmatrix} = \begin{pmatrix} -\gamma B_{1Y} m_z \\ 0 \\ +\gamma B_{1Y} m_X \end{pmatrix}$$

- Y component : $m_Y(t) = 0$
  $(\because\ m_Y(0) = 0)$
- X,z component : ?

- Case of $\boldsymbol{B} = B_0 \boldsymbol{e}_z$

$$\begin{pmatrix} \frac{dm_x}{dt} \\ \frac{dm_y}{dt} \\ \frac{dm_z}{dt} \end{pmatrix} = \begin{pmatrix} +\gamma B_0 m_y \\ -\gamma B_0 m_x \\ 0 \end{pmatrix}$$

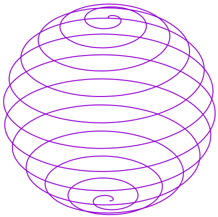
  - Clockwise rotation around z axis.
  - frequency: $\omega_0 = \gamma B_0$

- In the case of $\boldsymbol{B} = B_0 \boldsymbol{e}_z + B_{1Y} \boldsymbol{e}_Y$

$$\begin{pmatrix} \frac{dm_X}{dt} \\ \frac{dm_Y}{dt} \\ \frac{dm_z}{dt} \end{pmatrix} = \begin{pmatrix} -\gamma B_{1Y} m_z \\ 0 \\ +\gamma B_{1Y} m_X \end{pmatrix}$$

  - Clockwise rotation around Y axis.
  - frequency:$\omega_{1Y} = \gamma B_{1Y}$

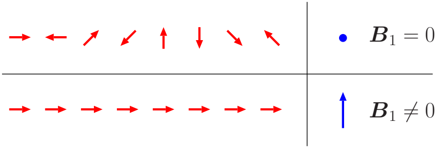
- In the case of $\boldsymbol{B} = B_0 \boldsymbol{e}_z + B_{1Y} \boldsymbol{e}_Y$
  While $\boldsymbol{m}$ rotates around z axis caused by $\boldsymbol{B}_0$, $\boldsymbol{m}$ always feels $\boldsymbol{B}_{1Y}$.
  As the result, $\boldsymbol{m}$ also rotates around Y axis.



Phase difference between $\boldsymbol{m}$ and $\boldsymbol{B}_1$

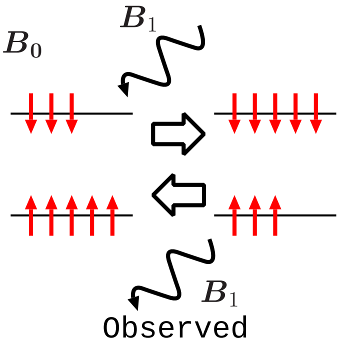
- When $\boldsymbol{B}_0$ is supplied, the precession is appeared. However, all the dipoles do not have same phase.
  $\rightarrow$ $\boldsymbol{e}_X$ of each particle is different.
  $$\boldsymbol{e}_X^{(i)} \neq \boldsymbol{e}_X^{(i')}, \quad \boldsymbol{m}^{(i)} \cdot \boldsymbol{B}_1 \neq 0$$
- Case of $B_{1X}^{(i)} \neq 0$
  $$\boldsymbol{B}_1(t) = B_{1X}^{(i)} \boldsymbol{e}_X^{(i)}(t) + B_{1Y}^{(i)} \boldsymbol{e}_Y^{(i)}(t)$$
  - torque owing to $B_{1X}^{(i)}$:
    $$\boldsymbol{m}^{(i)} \times B_{1X}^{(i)} \boldsymbol{e}_X^{(i)} = m_z^{(i)} B_{1X}^{(i)} \boldsymbol{e}_Y^{(i)} - m_Y^{(i)} B_{1X}^{(i)} \boldsymbol{e}_z$$
    $$\rightarrow \frac{dm_Y^{(i)}}{dt} = \gamma m_z^{(i)} B_{1X}^{(i)} \neq 0$$
  - $\boldsymbol{m}^{(i)}$ rotates so as it becomes perpendicular to $\boldsymbol{B}_1$.
    $(|\boldsymbol{m}| = \mathrm{const})$

- When $\boldsymbol{B} = \boldsymbol{B}_0$, directions of X axis for individual particles are different.
  $$\boldsymbol{m}^{(i)} = {m_X}^{(i)} \boldsymbol{e}_X^{(i)} + m_z^{(i)} \boldsymbol{e}_z$$
  But average has only z component.
  $$\boldsymbol{M} = \overline{\boldsymbol{m}} = \overline{m_z} \boldsymbol{e}_z$$
- When $\boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{B}_1$, X axis for all particles points to the direction which is perpendicular to $\boldsymbol{B}_1$.
  $\boldsymbol{m}^{(i')} = {m_X}^{(i)} \boldsymbol{e}_X + m_z^{(i)} \boldsymbol{e}_z$ Average has X-z component.
  $$\boldsymbol{M}' = \overline{\boldsymbol{m}'} = \overline{m_X} \boldsymbol{e}_X + \overline{m_z} \boldsymbol{e}_z$$



Excitation and Radiation

- Excitation
  Iradiate $\boldsymbol{B}_1$ to the state of $\boldsymbol{B} = \boldsymbol{B}_0$
  $\downarrow$
  Direction of $\boldsymbol{M}$ varies by absorption of energy of $\boldsymbol{B}_1$.
- Emission
  Stop excitation
  $\downarrow$
  While returning back to the original state, Energy of $\boldsymbol{m}$ is radiated as $\boldsymbol{B}_1$.



10.4 Relaxation time

- After excitation field, return back to Boltzmann distribution caused by interaction with neighboring.
  Electromagnetic wave is radiated during this process.

- Relaxation model
  - Longitudinal (Spin-lattice) relaxation

$$\frac{dM_z}{dt} = -\frac{M_z - M_0}{T_1}$$

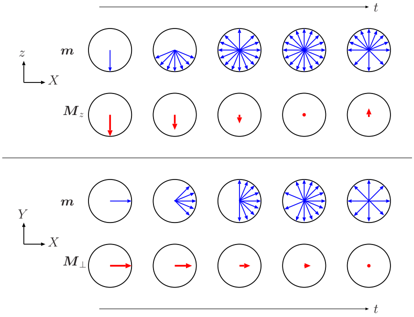
$$M_z(t) = M_0(1 - e^{-\frac{t}{T_1}}) + M_z(0)e^{-\frac{t}{T_1}}$$

  - Transverse (Spin-spin) relaxation

$$\frac{dM_\perp}{dt} = -\frac{M_\perp}{T_2}$$

$$M_\perp(t) = M_\perp(0)e^{-\frac{t}{T_2}}$$

- T_1, T_2 depends on both nuclear species and bonding to others



10.5 Principle of reconstruction
(1) Selection of excited slice

- Add gradient of magnetic field $G_z \equiv \boldsymbol{e}_z \cdot \nabla(\boldsymbol{B} \cdot \boldsymbol{e}_z)$

$$\boldsymbol{B}_z(z) = (B_0 + G_z(z - z_0))\boldsymbol{e}_z$$
$$\omega(z) = \omega_0 + \gamma G_z(z - z_0)$$

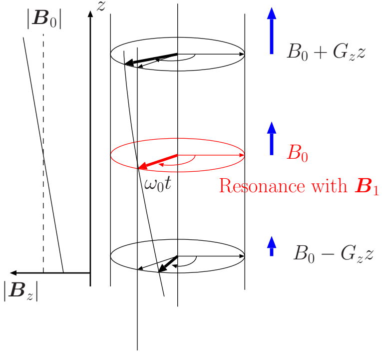
The frequency of precession depends on z.

- Excitation by the rotational magnetic field $\boldsymbol{B}_1(t) \propto \cos(\omega_0 t)$

$\downarrow$

Only the dipoles at $z = z_0$ are excited.

Others are not Excited.

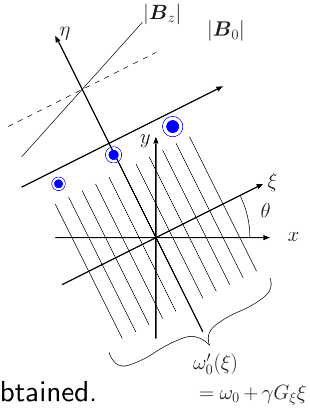


(2) Identification of line integrals of radiated field

- Add gradient of magnetic field on (x, y) plane, after stop irradiation of excited wave. $G_\xi \equiv \boldsymbol{e}_\xi \cdot \nabla(\boldsymbol{B}_z \cdot \boldsymbol{e}_z)$
  $\boldsymbol{B}_z(\xi) = (B_0 + G_\xi \xi)\boldsymbol{e}_z, \quad (\boldsymbol{e}_\xi = \cos\theta \boldsymbol{e}_x + \sin\theta \boldsymbol{e}_y)$

  The frequency of radiated wave depends on ξ.
  $$\omega_0'(\xi) = \gamma(B_0 + G_\xi \xi) = \omega_0 + \gamma G_\xi \xi$$

- Observed signal :
  Proportional to density of excited atom, ρ.
  $$s(t, \theta) = \iint \rho(x, y) e^{j\omega_0'(\xi(x,y))t}\, dx\, dy$$
  $\downarrow \mathcal{F}_t$ (shown in the next slide)
  $$S(\omega, \theta) = \int_{-\infty}^{\infty} \rho(\xi(\omega), \theta)\, d\eta$$
  $\equiv$projection $\left(\xi(\omega) = \frac{\omega - \omega_0}{\gamma G_\xi}\right)$



- Change the direction of gradient
  Projection data from whole direction can be obtained.
  $\rightarrow$ Same to the CT reconstruction.

Calculation of Fourier Transform of $s(t,\theta)$

$$\boxed{\begin{gathered}\xi \equiv \xi(x,y;\theta), \quad \eta \equiv \eta(x,y;\theta) \\ \omega_0'(\xi) = \omega_0 + \gamma G_\xi \xi \equiv \omega_0'(x,y,\theta)\end{gathered}}$$

$$\begin{aligned}
s(t,\theta) &= \int_y \int_x \rho(x,y) e^{j\omega_0' t}\, dx\, dy = \int_\eta \int_\xi \rho(\xi,\eta) e^{j\omega_0' t}\, d\xi\, d\eta \\
S(\omega,\theta) &= \int s(t,\theta) e^{-j\omega t}\, dt = \int_t \int_\eta \int_\xi \rho(\xi,\eta) e^{j\left(\omega_0 + \gamma G_\xi \xi - \omega\right) t}\, d\xi\, d\eta\, dt \\
&= \int_\eta \int_\xi \rho(\xi,\eta) \int_t e^{j\left(\omega_0 + \gamma G_\xi \xi - \omega\right) t}\, dt\, d\xi\, d\eta \\
&= \int_\eta \int_\xi \rho(\xi,\eta) \delta(\omega_0 + \gamma G_\xi \xi - \omega)\, d\xi\, d\eta \\
&= \int_{L_\eta} \rho(\xi,\eta)\, d\eta \quad \left(L_\eta \in \left\{ \boldsymbol{r}(\xi,\eta;\theta) \,\middle|\, \xi = \frac{\omega - \omega_0}{\gamma G_\xi} \right\} \right)
\end{aligned}$$

$$\boxed{S(\omega,\theta) = \int_{-\infty}^{\infty} \rho(\xi(\omega),\theta)\, d\eta, \quad \xi(\omega) = \frac{\omega - \omega_0}{\gamma G_\xi}}$$

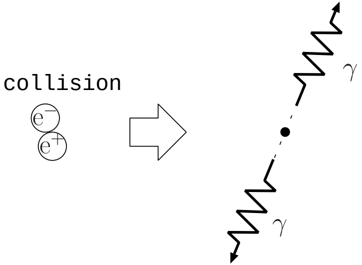
11. PET, SPECT

PET and SPECT can determine γ-ray source distribution to monitor level of biological activity.
e.g. Giving the drug which can emit γ-ray to blood vessel, we can observe amounts of blood flow.

- PET : Positron Emission Tomography
  The drug is source of positrons.
  The γ-rays are emitted by an annihilation of electron pair.
- SPECT : Single Photon Emission Computed Tomography
  The drug is source of γ-rays.

11.1 PET(Positron Emission Tomography)

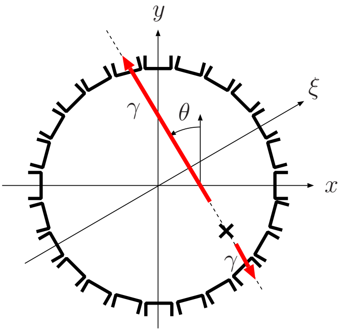
Annihilation of electron pair



$$\mathrm{e}^- + \mathrm{e}^+ \to 2\gamma(511\mathrm{keV}), \quad m_e c^2 = 511\mathrm{keV}$$

When an electron and a positron collide, two gamma-rays with 511 keV are emitted to opposite directions on a single line.
The direction of the line is random.

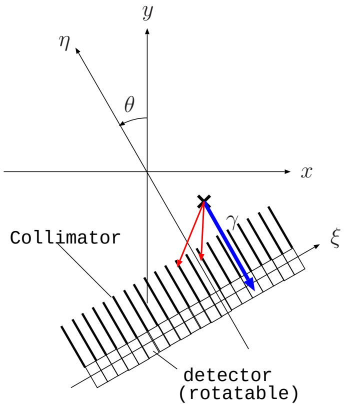
Projection data



- When annihilation occurs, two sensors located on the line passing the annihilation point detect an event at same time.
- The angle of the line corresponds to the projection angle θ in CT.
- The distance of the line from origin corresponds to the position of detector ξ in CT.
- Accumulating other events, we can obtain the statistical distribution $p(\xi, \theta)$.

The procedure to compute internal field $f(x, y)$ is same to CT.

11.2 SPCET(Single Photon Emission Computed Tomography)

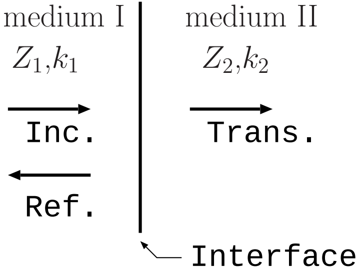


- When the array detector is aligned to perpendicular to the direction of θ, only the γ-ray with the angle θ can be detected by the detector located at ξ, since the others are shielded by the collimator.
- The projection of $p(\xi, \theta)$ corresponds to the accumulated events.

The procedure to compute internal field $f(x, y)$ is same to CT.

12. Ultrasonic Echo
12.1 Mechanism of echo

Reflection at interface

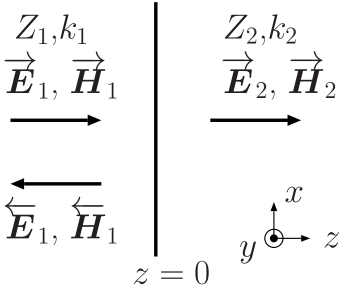


Z_i: Impedance
k_i: Wave number

The reflection occurs at the interface with impedance change.

(Proof is shown later.)

Reflection model for the electromagnetic wave



(Arrows over symbols represent propagating directions.)

- Poynting vector:

$$\overrightarrow{\boldsymbol{s}_i} = \overrightarrow{\boldsymbol{E}_i} \times \overrightarrow{\boldsymbol{H}_i} = \overrightarrow{s_i}\,\boldsymbol{e}_z \qquad (1)$$

$$\overleftarrow{\boldsymbol{s}_i} = \overleftarrow{\boldsymbol{E}_i} \times \overleftarrow{\boldsymbol{H}_i} = \overleftarrow{s_i}\,(-\boldsymbol{e}_z) \qquad (2)$$

- Definition of EM-wave:

(Definition to satisfy Eq. (1) and (2))

$$\begin{cases} \overrightarrow{\boldsymbol{E}_i} = +\overrightarrow{E_i}e^{-jk_iz}\boldsymbol{e}_x \\ \overrightarrow{\boldsymbol{H}_i} = +\overrightarrow{H_i}e^{-jk_iz}\boldsymbol{e}_y \\ \overleftarrow{\boldsymbol{E}_i} = +\overleftarrow{E_i}e^{+jk_iz}\boldsymbol{e}_x \\ \overleftarrow{\boldsymbol{H}_i} = -\overleftarrow{H_i}e^{+jk_iz}\boldsymbol{e}_y \end{cases} \qquad (3)$$

- Definition of impedance:

$$Z_i = \frac{\overrightarrow{E_i}}{\overrightarrow{H_i}} = \frac{\overleftarrow{E_i}}{\overleftarrow{H_i}} \qquad (4)$$

- Field in each medium

$$\begin{cases} \boldsymbol{E}_1 = \overrightarrow{\boldsymbol{E}_1} + \overleftarrow{\boldsymbol{E}_1}, \quad \boldsymbol{E}_2 = \overrightarrow{\boldsymbol{E}_2}, \\ \boldsymbol{H}_1 = \overrightarrow{\boldsymbol{H}_1} + \overleftarrow{\boldsymbol{H}_1}, \; \boldsymbol{H}_2 = \overrightarrow{\boldsymbol{H}_2} \end{cases} \tag{5}$$

- Boundary condition: (Continuity of tangential components)
  At $z = 0$

$$\begin{cases} \boldsymbol{E}_1 \cdot \boldsymbol{e}_x = \boldsymbol{E}_2 \cdot \boldsymbol{e}_x \\ \boldsymbol{H}_1 \cdot \boldsymbol{e}_y = \boldsymbol{H}_2 \cdot \boldsymbol{e}_y \end{cases} \tag{6}$$

$$\Leftrightarrow \begin{cases} \overrightarrow{E_1} + \overleftarrow{E_1} = \overrightarrow{E_2} \\ \overrightarrow{H_1} - \overleftarrow{H_1} = \overrightarrow{H_2} \end{cases} \tag{7}$$

From Eq. (4) and Eq. (7)

- Reflection wave

$$\overleftarrow{E_1} = \underbrace{\frac{Z_2 - Z_1}{Z_2 + Z_1}}_{\equiv R} \overrightarrow{E_1} \tag{8}$$

- Transmission wave

$$\overrightarrow{E_2} = \frac{2Z_2}{Z_2 + Z_1} \overrightarrow{E_1} \tag{9}$$

In the case $Z_2 \neq Z_1$:

$$R = \frac{Z_2 - Z_1}{Z_2 + Z_1} \neq 0.$$

$\Rightarrow$ Reflection occurs at the interface.

12.2 Impedance

Electro-Magnetic wave	$Z = \dfrac{E\ [\mathrm{V/m}]}{H\ [\mathrm{A/m}]}$
Electronics	$Z = \dfrac{V\ [\mathrm{V}]}{I\ [\mathrm{A}]} = \dfrac{\text{Electric potential}}{\text{Electric current } (\propto \text{ velocity})}$
Fluid mechanics	$Z = \dfrac{p}{v} = \dfrac{\text{Acoustic pressure (potential)}}{\text{Acoustic velocity}}$

Impedance of fluid depends on the velocity.

Acoustic velocity and Frequency

- Acoustic velocity

Medium	Acoustic velocity [m/s]
Air	344
Water	$\simeq$1,500
Fat	$\simeq$ 1,450
internal organs, muscle	$\simeq$ 1,550

- Frequency(at $v = 1,500$)

  $f = v/\lambda$

  λ 0.5 mm $\sim$ 0.1 mm

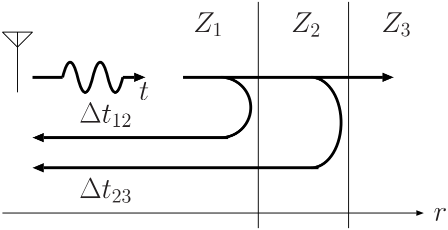
  f 3 MHz $\sim$ 15 MHz

  $\rightarrow$ Ultrasonic wave

12.3 Measurement system

Similarly to lader, signals are measured in polar coordinate (range and direction).

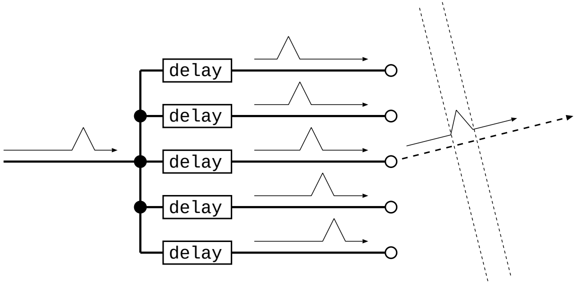
Range measurement



1. Emit the pulse modulated acoustic wave
2. Measure time delay of reflected pulse
3. Calculate $Z(r)$

Scanning the direction

Emitting direction can be controlled by phased array antenna:



(Non-plane wave can be formed by controlling the delays.)