2. パワースペクトルと相関関数

- 相関とは?
  離散的な二つの変量の関係 (x_i, y_i)
- 相関関数とは?
  変量が連続的な変化とみなせる場合の複数の変量の関係 $(x(t), y(t))$
- パワースペクトルとは?
  変量 $(x(t))$ のフーリエ変換 $(X(\omega))$ の強さを表す指標
- Wiener-Khintchine の定理
  パワースペクトルと自己相関関数の関係

2.1 パワースペクトルの定義

フーリエ変換：

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega \qquad (1)$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt \qquad (2)$$

パワースペクトル密度 の定義

$$S(\omega) = \lim_{T\to\infty}\frac{|X(\omega)|^2}{T} \qquad (3)$$

- 信号の周波数 ω 成分の測定時間 T に関する振幅自乗の平均
- 位相情報は持たない
- $S(\omega)\,d\omega$ がパワースペクトルを表す
- $|X(\omega)|^2$ をエネルギースペクトル密度と呼ぶ
- 次元解析:

FT of x	:	$[X] = [x \cdot t]$
Power spectral density	:	$[S] = \left[\frac{\lvert X\rvert^2}{T}\right] = \left[x^2 \cdot t\right]$
Power spectrum	:	$[S\,d\omega] = \left[\frac{X^2}{t}\frac{1}{t}\right] = \left[x^2\right]$
Energy spectral density	:	$\left[\lvert X\rvert^2\right] = \left[x^2 t^2\right]$

2.2 相関

母集団: サンプル f_i $(i \in (1, \cdots, N))$

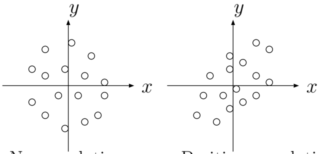
平均 $$\mathrm{E}\left[f\right] \equiv \frac{1}{N}\sum_{i=1}^{N} f_i \qquad (4)$$

分散 $$\sigma_f^2 \equiv \mathrm{E}\left[(f - \mathrm{E}\left[f\right])^2\right] \qquad (5)$$

相関: 二つの変量 (x_i, y_i) の類似度を表す指標

$$C = \mathrm{E}\left[x'y'\right], \quad \text{or} \quad r = \frac{\mathrm{E}\left[x'y'\right]}{\sqrt{\mathrm{E}\left[x'^2\right]\mathrm{E}\left[y'^2\right]}} \qquad (6)$$

$$(x'_i = x_i - \mathrm{E}\left[x\right], \quad y'_i = y_i - \mathrm{E}\left[y\right])$$



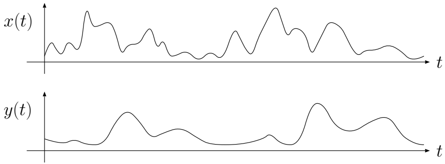
No correlation Positive correlation

2.3 相関関数

x, y が時系列データの場合

$$\mathrm{E}[x] \to \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\,dt = \langle x(t) \rangle_t \quad \text{(時間平均)} \tag{7}$$

e.g. $x(t)$: 降水量,
$y(t)$: 川の水量



- 時間遅れ
- 細かい変動の平滑化

相互相関関数と自己相関関数

相互相関関数 (cross-correlation function)

$$C_{xy}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) y(t+\tau)\, dt = \langle x(t) y(t+\tau) \rangle_t \tag{8}$$

一つの変量のみに注目した場合 ($y(t) = x(t)$) でも周期性が解る。

自己相関関数 (auto-correlation function)

$$C(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) x(t+\tau)\, dt = \langle x(t) x(t+\tau) \rangle_t \tag{9}$$

(規格化)

$$R(\tau) = \frac{C(\tau)}{C(0)} = \frac{\langle x(t) x(t+\tau) \rangle_t}{\langle x(t)^2 \rangle_t}, \qquad R(0) = 1$$

$\rightarrow$ 自己相関係数

周期関数の自己相関関数

(e.g. 1) $x(t) = A\cos\omega_1 t$

$$C(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}(A\cos\omega_1 t)(A\cos\omega_1(t+\tau))\,dt$$

$$= A^2\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\left(\cos^2\omega_1 t\cos\omega_1\tau - \cos\omega_1 t\sin\omega_1 t\sin\omega_1\tau\right)\,dt$$

$$= A^2\lim_{T\to\infty}\Bigg[\underbrace{\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\frac{1+\cos 2\omega_1 t}{2}\,dt}_{=\frac{1}{2}}\cos\omega_1\tau - \underbrace{\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\frac{\sin 2\omega_1 t}{2}\,dt}_{=0}\sin\omega_1\tau\Bigg] = \frac{A^2}{2}\cos\omega_1\tau$$

$$\underline{R(\tau) = \cos\omega_1\tau}$$

(e.g. 2) $x(t) = A\sin\omega_1 t = A\cos\left(\omega_1 t - \frac{\pi}{2}\right)$

$$C(\tau) = A^2\lim_{T\to\infty}\left[\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\frac{1+\cos\left(2\left(\omega_1 t-\frac{\pi}{2}\right)\right)}{2}\,dt\cos\omega_1\tau - \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\frac{\sin\left(2\left(\omega_1 t-\frac{\pi}{2}\right)\right)}{2}\,dt\sin\omega_1\tau\right] = \frac{A^2}{2}\cos\omega_1\tau$$

$$\underline{R(\tau) = \cos\omega_1\tau}$$

- 周期性が現れる。
- 位相には依存しない。$\rightarrow$ t の原点には依存しない。

自己相関関数の性質

- 原点に依存しない (証明略)
- 偶関数性 ($C(-\tau) = C(\tau)$)

$$\left(\begin{aligned} C(-\tau) &= \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)x(t-\tau)\,dt \qquad (t' = t-\tau) \\ &= \lim_{T\to\infty} \int_{-\frac{T}{2}-\tau}^{\frac{T}{2}-\tau} x(t'+\tau)x(t')\,dt' \qquad \left(\frac{T}{2} \gg |\tau|\right) \\ &= \lim_{T\to\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t'+\tau)x(t')\,dt' = C(\tau) \end{aligned}\right)$$

- $\tau = 0$ で最大

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} (x(t) \pm x(t+\tau))^2 \; dt \geq 0$$

$$\left(\begin{aligned} \text{LHS} &= \underbrace{\int x^2(t)\,dt}_{C(0)\geq 0} + \underbrace{\int x^2(t+\tau)\,dt}_{C(0)\geq 0} \pm 2 \underbrace{\int x(t)x(t+\tau)\,dt}_{C(\tau)} \\ &= 2(C(0) \pm C(\tau)) \geq 0 = \text{RHS} \\ &\therefore C(0) \geq |C(\tau)| \end{aligned}\right)$$

- $C(\tau)$ の1階微分

$$\left(\begin{aligned}
&C'(\tau) = \frac{dC}{d\tau} = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\underbrace{\frac{\partial x(t+\tau)}{\partial\tau}}_{=x'(t+\tau)}\,dt = \langle x(t)x'(t+\tau)\rangle_t \\
&\text{(Replacing } t+\tau=\xi \to dt = d\xi\text{, because } \tfrac{\partial(t+\tau)}{\partial\tau}=1) \\
&\qquad = \lim_{T\to\infty}\int_{-\frac{T}{2}+\tau}^{\frac{T}{2}+\tau} x(\xi-\tau)x'(\xi)\,d\xi \\
&\qquad = \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(\xi-\tau)x'(\xi)\,d\xi = \langle x(t-\tau)x'(t)\rangle_t
\end{aligned}\right)$$

$x(t)$ と $x'(t)$ の
相互相関関数

- $C(\tau)$ の2階微分

$$\left(\begin{aligned}
&C''(\tau) = \frac{d^2C}{d\tau^2} = \langle x(t)x''(t+\tau)\rangle_t \\
&\qquad \text{(または } t-\tau=\eta \text{ と置けば } (\tfrac{\partial(t-\tau)}{\partial\tau}=-1)) \\
&\qquad = \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} \left(-x'(\eta)x'(\eta+\tau)\right)\,d\eta \\
&\qquad = -\langle x'(t)x'(t+\tau)\rangle_t
\end{aligned}\right)$$

$x(t)$ と $x''(t)$ の
相互相関関数

かつ

$x'(t)$ の
自己相関関数の負

2.4 パワースペクトルと自己相関関数の関係

信号の定義域: $(x(t) \in \mathbb{R})$

$$x(t) : \begin{cases} \neq 0 & -\frac{T}{2} \leq t \leq \frac{T}{2} \\ = 0 & \text{elsewhere} \end{cases}$$

フーリエ積分:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega \left(= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega)e^{-j\omega t}\,d\omega \right)$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

自己相関関数:

$$\begin{aligned}
C(\tau) &= \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)x(t+\tau)\,dt \\
&= \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega')e^{-j\omega' t}\,d\omega'\right]\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega(t+\tau)}\,d\omega\right] dt \\
&= \lim_{T\to\infty}\frac{1}{2\pi T}\int_{-\infty}^{\infty} X(\omega)e^{j\omega\tau}\int_{-\infty}^{\infty} X^*(\omega')\underbrace{\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j(\omega-\omega')t}\,dt\right)}_{\delta(\omega-\omega')} d\omega'\,d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{\lim_{T\to\infty}\frac{X^*(\omega)X(\omega)}{T}}_{=S(\omega)} e^{j\omega\tau}\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)e^{j\omega\tau}\,d\omega
\end{aligned}$$

$C(\tau)$ は $S(\omega)$ の逆 FT になっている。

Wiener-Khintchine の定理

$$C(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)e^{j\omega\tau}\,d\omega \quad (10)$$

$$S(\omega) = \int_{-\infty}^{\infty} C(\tau)e^{-j\omega\tau}\,d\tau \quad (11)$$

$$\begin{array}{ccc} x(t) & \xrightarrow{\langle x(t)x(t+\tau)\rangle_t} & C(\tau) \\ \downarrow\uparrow \text{FT} & & \downarrow\uparrow \text{FT} \\ X(\omega) & \xrightarrow{\langle X(\omega)X^*(\omega)\rangle_t} & S(\omega) \end{array}$$

$\langle\cdots\rangle_t$: 観測時間 $T\to\infty$ の場合の平均

2.5 クロススペクトルと相互相関関数

$$C_{xy}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xy}(\omega)e^{j\omega\tau}\,d\omega \quad (12)$$

$$S_{xy}(\omega) = \int_{-\infty}^{\infty} C_{xy}(\tau)e^{-j\omega\tau}\,d\tau \quad (13)$$

クロススペクトル密度

$$\begin{aligned} S_{xy}(\omega) &= \lim_{T\to\infty}\frac{X^*(\omega)Y(\omega)}{T} \\ &= \langle X^*(\omega)Y(\omega)\rangle_t \end{aligned} \quad (14)$$

$$\left(\begin{aligned} C_{xy}(\tau) &= \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)y(t+\tau)\,dt = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x^*(t)y(t+\tau)\,dt \qquad (\because x(t), y(t)\in\mathbb{R}) \\ &= \lim_{T\to\infty}\frac{1}{T}\int_{-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{\omega} X^*(\omega)e^{-j\omega t}\,d\omega\right)\left(\frac{1}{2\pi}\int_{\omega'} Y(\omega)e^{j\omega'(t+\tau)}\,d\omega'\right)dt \\ &= \lim_{T\to\infty}\frac{1}{2\pi T}\int_{\omega} X^*(\omega)e^{j\omega\tau}\int_{\omega'} Y(\omega')\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j(\omega'-\omega)t}\,dt\right)d\omega'\,d\omega \\ &= \lim_{T\to\infty}\frac{1}{2\pi T}\int_{\omega} X^*(\omega)e^{j\omega\tau}\int_{\omega'} Y(\omega')\delta(\omega'-\omega)\,d\omega'\,d\omega = \lim_{T\to\infty}\frac{1}{2\pi T}\int_{\omega} X^*(\omega)Y(\omega)e^{j\omega\tau}\,d\omega \\ &= \frac{1}{2\pi}\int_{\omega}\lim_{T\to\infty}\frac{X^*(\omega)Y(\omega)}{T}e^{j\omega\tau}\,d\omega = \frac{1}{2\pi}\int_{\omega} S_{xy}(\omega)e^{j\omega\tau}\,d\omega \end{aligned}\right)$$

クロススペクトルの性質

$$C_{xy}(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)y(t+\tau)\,dt = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} y(t)x(t-\tau)\,dt = C_{yx}(-\tau)$$

$$\to C_{xy}(\tau) \in \mathbb{R} \qquad (\because x(t), y(t) \in \mathbb{R})$$

$$S_{xy}(\omega) = \mathcal{F}\left\{C_{xy}(\tau)\right\}$$

$$C_{xy}(-\tau) = C_{yx}(\tau) \tag{15}$$

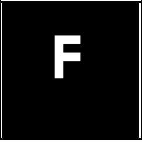
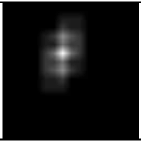
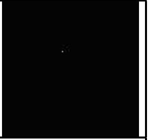
$$S_{xy}(-\omega) = S_{xy}^{*}(\omega) \tag{16}$$

$$S_{yx}(\omega) = S_{xy}^{*}(\omega) \tag{17}$$

$$S_{xy}(-\omega) = S_{yx}(\omega) \tag{18}$$

移動量評価

$g(\boldsymbol{r}) = f(\boldsymbol{r}+\boldsymbol{\Delta}),\quad \boldsymbol{\Delta} = (-8, 16)$ (Image size:128x128)

$f(\boldsymbol{r})$	$g(\boldsymbol{r})$	$C_{fg}(\boldsymbol{r})$ (w/o POC)	$\widehat{C}_{fg}(\boldsymbol{r})$ (w/ POC)
			
		$(-8, 16)$	$(-8.0, 16.0)$

$$\left(\begin{aligned} C_{fg} &= \mathcal{F}^{-1}\left\{\langle F^* G\rangle_t\right\} \\ \widehat{C}_{fg} &= \mathcal{F}^{-1}\left\{\left\langle \widehat{F}^* \widehat{G}\right\rangle_t\right\} \\ \widehat{F} &= \frac{F}{|F|},\quad \widehat{G} = \frac{G}{|G|} \\ \widehat{F}^*\widehat{G} &= e^{i(\phi_G - \phi_F)} \end{aligned}\right)$$

POC:Phase Only cross-Correlation

$g(\boldsymbol{r}) = f(\boldsymbol{r}+\boldsymbol{\Delta})$ の場合

$$G(\boldsymbol{k}) = e^{i\boldsymbol{k}\cdot\boldsymbol{\Delta}} F(\boldsymbol{k})$$

$$\widehat{F}^*(\boldsymbol{k})\widehat{G}(\boldsymbol{k}) = e^{i\boldsymbol{k}\cdot\boldsymbol{\Delta}}$$

($\phi_F(\boldsymbol{k})$ is canceled.)

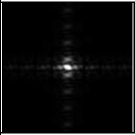
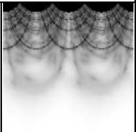
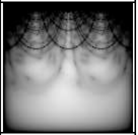
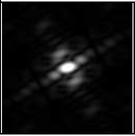
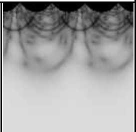
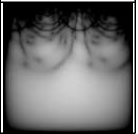
$$\begin{aligned} \widehat{C}_{fg}(\boldsymbol{r}) &= \mathcal{F}^{-1}\left\{\left\langle e^{i\boldsymbol{k}\cdot\boldsymbol{\Delta}}\right\rangle_r\right\} \\ &= \lim_{L\to\infty}\frac{1}{L^2}\frac{1}{(2\pi)^2}\int e^{i\boldsymbol{k}\cdot(\boldsymbol{\Delta}-\boldsymbol{r})}\,d\boldsymbol{k}^2 \\ &= \lim_{L\to\infty}\frac{1}{L^2}\delta(\boldsymbol{r}-\boldsymbol{\Delta}) \to \boxed{\text{鋭いピーク}} \end{aligned}$$

回転・拡大率評価

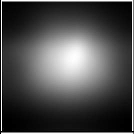
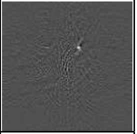
$$g(\boldsymbol{r}) = f(\alpha \boldsymbol{\Theta} \cdot \boldsymbol{r} + \boldsymbol{\Delta}), \quad \boldsymbol{\Delta} = (-8, 16),$$

$$\alpha = \frac{1}{2}, \ \boldsymbol{\Theta} = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix}, \ \theta_1 = 30\,\mathrm{deg}$$

(Image size:128x128)

(x, y)	Cartesian (k_x, k_y)	log-polar $(k_\theta, \log\lvert\boldsymbol{k}\rvert)$	Hanning-Win. $(k_\theta, \log\lvert\boldsymbol{k}\rvert)$
$f(\boldsymbol{r})$	$I_F(\boldsymbol{k}) = \lvert F(\boldsymbol{k})\rvert$		$\mathcal{H}\{I_F(\boldsymbol{k})\}$
			
$g(\boldsymbol{r})$	$I_G(\boldsymbol{k}) = \lvert G(\boldsymbol{k})\rvert$		$\mathcal{H}\{I_G(\boldsymbol{k})\}$
			
		(log scale $[0.05, 500]$)	(log scale $[0.05, 500]$)

$$C_{I_F I_G} = \mathcal{F}^{-1}\{(\mathcal{F}\{\mathcal{H}\{I_F\}\})^* \mathcal{F}\{\mathcal{H}\{I_G\}\}\}$$

	Cross-corr. of I_F and I_G $(k_\theta, \log\lvert\boldsymbol{k}\rvert)$ $C_{I_F I_G}(\boldsymbol{k})$	
	w/o POC	w/ POC
		
k_θ [deg]	19.6	29.4
$\log\lvert\boldsymbol{k}\rvert$	-0.343	-0.682
$\frac{1}{\lvert\boldsymbol{k}\rvert}$ [times]	0.710	0.505