8.2 Wiener deconvolution

- **観測モデル**：

$$y(t) = h(t) * \widehat{x}(t) + n(t) \tag{1}$$

$(y(t), h(t)$: known$)$

$$Y(f) = H(f)\cdot \widehat{X}(f) + N(f) \tag{2}$$

$$\left(\widehat{X} = \frac{Y-N}{H}\right)$$

$$|N(f)| \sim |N'(f)| \quad (|N'(f)| : \text{known}) \tag{3}$$

- **推定値** $\widetilde{x}$：

$$\widetilde{X}(f) = \Psi_x(f)Y(f) \qquad (\Psi_x \in \mathbb{C}) \tag{4}$$

$$\text{minimize } E = \int \underbrace{\left|\widetilde{X} - \widehat{X}\right|^2}_{=I} df \tag{5}$$

$$\rightarrow \frac{\partial E}{\partial \Psi_x} = 0 \tag{6}$$

$$\begin{aligned}
I &= \left|\widetilde{X} - \widehat{X}\right|^2 = \left|\Psi_x Y - \tfrac{Y-N}{H}\right|^2 \\
&= \left|\left(\Psi_x - \tfrac{1}{H}\right) Y + \tfrac{N}{H}\right|^2 \\
&= \left|\left(\Psi_x - \tfrac{1}{H}\right) Y\right|^2 + \tfrac{|N|^2}{|H|^2} \\
&\quad + \left(\Psi_x - \tfrac{1}{H}\right) Y \tfrac{N^*}{H^*} + \left(\Psi_x^* - \tfrac{1}{H^*}\right) Y^* \tfrac{N}{H}
\end{aligned}$$

$$\left(\begin{aligned}
&\left(\Psi_x - \tfrac{1}{H}\right) Y \tfrac{N^*}{H^*} \qquad \text{(Eq.(2))} \\
&= \left(\Psi_x - \tfrac{1}{H}\right)\left(H\widehat{X} + N\right)\tfrac{N^*}{H^*} \\
&= \underbrace{\left(\Psi_x - \tfrac{1}{H}\right)\tfrac{H}{H^*}\widehat{X}N^*}_{\text{積分}=0} + \left(\Psi_x - \tfrac{1}{H}\right)\tfrac{1}{H^*}|N|^2
\end{aligned}\right)$$

$$E = \int I(f)\, df = \int I'(f)\, df$$

$$\begin{aligned}
I' &= \left|\left(\Psi_x - \frac{1}{H}\right) Y\right|^2 - \frac{|N|^2}{|H|^2} \\
&\quad + \Psi_x \frac{|N|^2}{H^*} + \Psi_x^* \frac{|N|^2}{H}
\end{aligned}$$

Wiener deconvolution

$$\frac{\partial E}{\partial \Psi_x} = 0 \text{ or } \frac{\partial E}{\partial \Psi_x^*} = 0$$

$$E = \int I'(f)\, df$$

$$I' = \left| \left(\Psi_x - \tfrac{1}{H} \right) Y \right|^2 - \tfrac{|N|^2}{|H|^2} + \Psi_x \tfrac{|N|^2}{H^*} + \Psi_x^* \tfrac{|N|^2}{H}$$

$$\left(\begin{array}{l} \frac{\partial}{\partial \Psi_x} \left(\left| \left(\Psi_x - \frac{1}{H} \right) Y \right|^2 \right) \\ \quad = \frac{\partial}{\partial \Psi_x} \left(\left(\Psi_x^* - \frac{1}{H^*} \right) \left(\Psi_x - \frac{1}{H} \right) |Y|^2 \right) \\ \quad = \left(\Psi_x^* - \frac{1}{H^*} \right) |Y|^2 \end{array} \right)$$

$$\frac{\partial I'}{\partial \Psi_x^*} = \left(\Psi_x - \frac{1}{H} \right) |Y|^2 + \frac{|N|^2}{H} = 0$$

$$\rightarrow \quad \Psi_x = \frac{1}{H} \underbrace{\frac{|Y|^2 - |N|^2}{|Y|^2}}_{=\Phi_y \text{ if } |N| = |N'|.}$$

Wiener deconvolution :

$$\Psi_x(f) = \frac{\Phi_y(f)}{H(f)} \quad (7)$$

$$\Phi_y(f) = \frac{|Y|^2 - |N'(f)|^2}{|Y(f)|^2} \quad (8)$$

$$\begin{aligned} \widetilde{X}(f) &= \Psi_x(f) Y(f) \\ &= \frac{1}{H(f)} \Phi_y(f) Y(f) \end{aligned} \quad (9)$$

Wiener deconvolution **を適用したスペクトルは，** $Y(f)$ **に** Wiener filter **を掛けた後に，応答関数** $H(f)$ **で割ったものと等価**

理想的 $(N'(f) = N(f))$ な場合

$$
\begin{aligned}
&(N' = N)\\
&\Psi_x = \tfrac{1}{H}\Phi_y\\
&\quad = \tfrac{1}{H}\tfrac{|Y|^2 - |N|^2}{|Y|^2} = \tfrac{1}{H}\tfrac{\left|\widehat{Y}\right|^2}{\left|\widehat{Y}\right|^2 + |N|^2}\\
&\quad = \frac{H^*\left|\widehat{X}\right|^2}{|H|^2\left|\widehat{X}\right|^2 + |N|^2}
\end{aligned} \tag{10}
$$

- $H \neq 0$ and $N = 0$:

  $\Psi_x = \frac{1}{H} \quad \rightarrow \quad \widetilde{X} = \frac{Y}{H} = \widehat{X}$

  (理想の状態)

- $H = 0$ and $N \neq 0$:

  $\Psi_x = 0 \quad \rightarrow \quad \widetilde{X} = 0$

  (復元不能)

- $H = 0$ and $N = 0$:

$$
\lim_{H \to 0}\left\{\lim_{N \to 0}\left|\frac{H^*\left|\widehat{X}\right|^2}{|H|^2\left|\widehat{X}\right|^2 + |N|^2}\right|\right\} = \lim_{H \to 0}\left\{\left|\frac{1}{H}\right|\right\} = \infty
$$

$$
\lim_{N \to 0}\left\{\lim_{H \to 0}\left|\frac{H^*\left|\widehat{X}\right|^2}{|H|^2\left|\widehat{X}\right|^2 + |N|^2}\right|\right\} = 0
$$

  定義不能
  (極限の取り方により異なる値に収束)

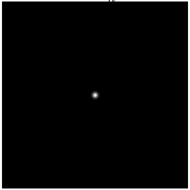
> $H = 0$ の場合，$\widetilde{X}$ は定まらない。
> $\rightarrow \left(\begin{array}{l} \widetilde{x}(t) \text{ を求めるために，} \\ \widetilde{X}(f) = 0 \text{ とみなす} \end{array} \right)$

Wiener Deconvolution の例（入力：$\sigma_h = 5, \sigma_n = 5$）

$\widehat{x}(\boldsymbol{r})$

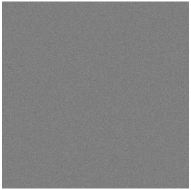
$[0, 255]$

$h(\boldsymbol{r}) = \frac{1}{2\pi\sigma_h^2} e^{-\frac{r^2}{2\sigma_h^2}}$



$[0, 1/(2\pi\sigma_h^2)]$

$n(\boldsymbol{r})$

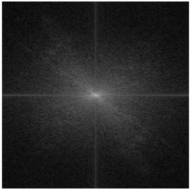


$[-64, 64]$

$y(\boldsymbol{r}) = h * \widehat{x} + n$

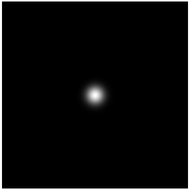
$[0, 255]$

$\widehat{X}(\boldsymbol{k})$



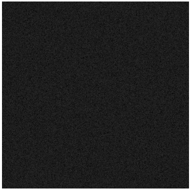
log $[1, 10^4]$

$H(\boldsymbol{k}) = e^{-\frac{\sigma_h^2 k^2}{2}}$



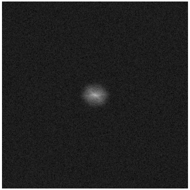
$[0, 1]$

$|N(\boldsymbol{k})|$



log $[1, 10^4]$

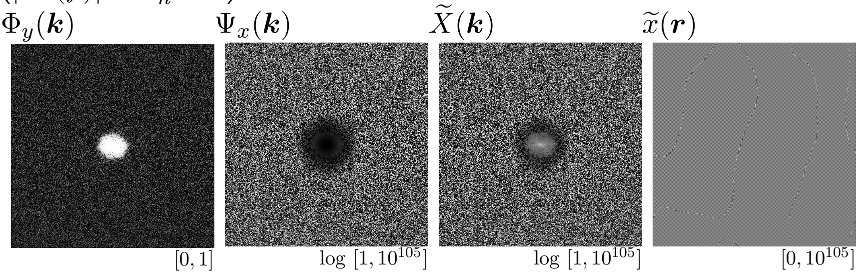
$Y(\boldsymbol{k}) = H\,\widehat{X} + N$



log $[1, 10^4]$

Wiener Deconvolution の例 (結果 (失敗例))

$(|N'(f)| = \sigma'_n = 5)$



$|\Psi_x(\boldsymbol{k})|$ に極端に大きな値が含まれるため，$\widetilde{X}(\boldsymbol{k})$ が発散し，$\widetilde{x}(\boldsymbol{r})$ も発散。

フィルターの発散の原因

- $H = 0$ and $N' \neq N$

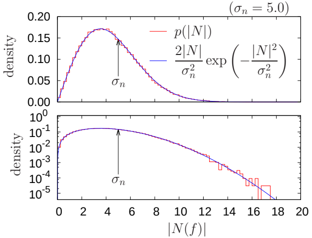
$$\Psi_x = \lim_{H\to 0} \frac{1}{H}\Phi_y$$
$$= \lim_{H\to 0} \frac{1}{H}\frac{\mathrm{Pos}\left\{|Y|^2 - |N'|^2\right\}}{|Y|^2}$$
$$= \lim_{H\to 0} \frac{1}{H}\frac{\mathrm{Pos}\left\{\left|H\widehat{X}+N\right|^2 - |N'|^2\right\}}{\left|H\widehat{X}+N\right|^2}$$
$$= \lim_{H\to 0} \frac{1}{H}\frac{\mathrm{Pos}\left\{|N|^2 - |N'|^2\right\}}{|N|^2}$$

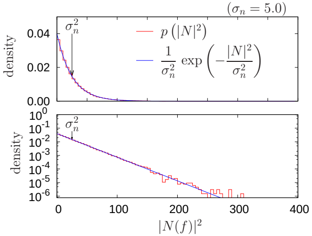
$$\mathrm{Pos}\left\{F\right\} \equiv \max\{F, 0\}$$

$$|\Psi_x| = \begin{cases} \infty & (|N| > |N'|) \\ 0 & (|N| \leq |N'|) \end{cases}$$

$\rightarrow$ 不連続

$N(f)$ の確率密度





フィルターの発散の抑止策

- $|N'|$ を大きめに選ぶ

$$\Psi'_x = \frac{1}{H} \frac{\mathrm{Pos}\left\{|Y|^2 - \alpha |N'|^2\right\}}{|Y|^2} \quad (\alpha > 1)$$

- 定義域の制限

$$\Psi'_x(\boldsymbol{k}) = \Theta\left\{|k| \le k_{\max}\right\} \Psi_x(\boldsymbol{k})$$

$$\left(\Theta\left\{\mathsf{C}\right\} \equiv \begin{cases} 1 & (\text{C is true}) \\ 0 & (\text{C is false}) \end{cases} \right)$$

- 値域の制限

$$\Psi'_x = \Theta\left\{|\Psi(k)| \le \Psi_{\max}\right\} \Psi_x(\boldsymbol{k})$$

- $|H|$ の制限

$$\Psi'_x = \Theta\left\{|H| > H_{\min}\right\} \Psi_x$$

- 近傍のチェック

$$\Psi'_x(\boldsymbol{k}) = \Theta\left\{\frac{M_0(\boldsymbol{k})}{M_a(\boldsymbol{k})} > r_{0_{\min}}\right\} \Psi_x(\boldsymbol{k})$$

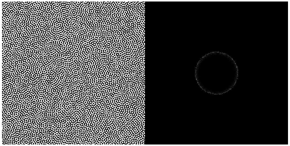
  - $M_a(\boldsymbol{k})$ は $\boldsymbol{k}$ の近傍の点 $\boldsymbol{k}'$ の数
  - $M_0(\boldsymbol{k})$ は近傍の点の内の $\Phi_y(\boldsymbol{k}') = 0$ の数

Wiener Deconvolution の発散抑止例

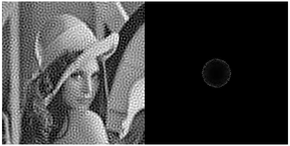
Before Deconv.

True

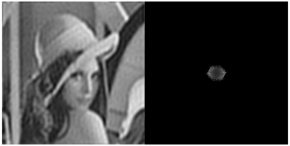
$k_{\max} = 0.15$: 発散



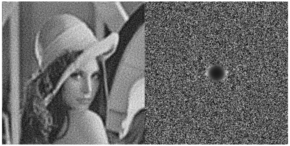
$k_{\max} = 0.10$: 周期的な模様



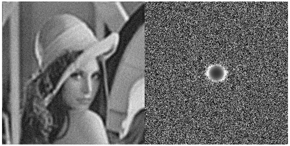
$\alpha = 4$: ぼけ



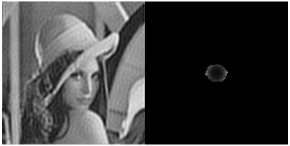
$\Psi_{\max} = 10$: Noisy



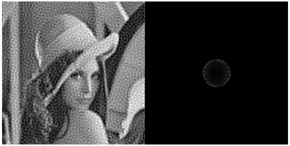
$\Psi_{\max} = 5$: Noisy



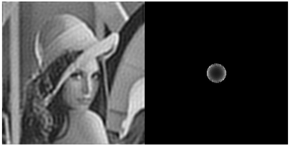
近傍チェック : Ringing



$H_{\min} = 0.01$: 周期模様



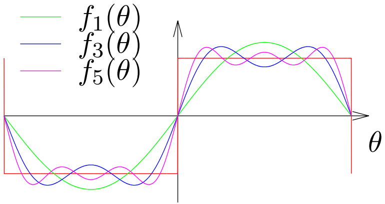
$H_{\min} = 0.1$: 周期模様, Ring.



Wiener deconvolution の誤差の原因

- 発散
  H が小さい場合に対する低減が不十分.
- ぼけ
  高周波成分のオーバーフィルタリング.
- 周期的な模様
  特定波長のフィルタリングが不十分.
- Noisy
  高周波成分のフィルタリングが不十分.

- Ringing
  エッジと平行にゴーストが発生.
  FT の有限項の打ち切りが原因.
  $\Leftrightarrow$ ギブス (Gibbs) 現象
  (Ringing を避けることはできない)



8.3 応答関数 $H(f)$ の推定

$$Y(f) = H(f) \cdot \widehat{X}(f) + N(f)$$

$$\widetilde{X}(f) = \frac{1}{H(f)} \Phi_y(f) Y(f)$$

$H(f)$: 関数形は解っているが，パラメータが未知

$$\left(\begin{array}{l} \text{e.g.: } H(f) = e^{-\frac{f^2}{2\sigma_f^2}} \\ \quad (\sigma_f \text{ is unknown}) \end{array} \right)$$

- 最小自乗法によりパラメータを決め，$H(f)$ の推定値 $\widetilde{H}(f)$ を求める。
- 最小自乗法を掛けるデータには，$N(f)$ が支配的な領域のノイズの確率密度関数を考慮して一様化したものを用いる。

$$\widetilde{X}'(f) = \frac{1}{\widetilde{H}(f)} \Phi_y(f) Y(f)$$

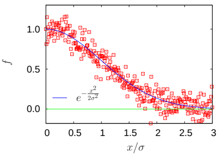
ガウス関数への最小自乗フィッティング

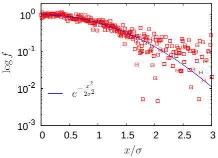
- フィッティング関数 (推定値)
  $$\widetilde{f}(x_i;\, a, b) = ae^{-bx_i^2}$$
- 観測値
  $$(x_i, f_i) \quad i \in \{1, \cdots, N\}$$
- 残差　$\Delta f_i = f_i - \widetilde{f}(x_i)$
- 残差自乗平均
  $$E = \overline{(\Delta f)^2}$$
- minimize E

- 一般的な方法では，正規方程式は，非線形連立方程式になる。
  $\rightarrow$ 複雑

- f から $F = \log f$ に写像
  $$\widetilde{F}_i(x_i; a, b) = \log a - bx_i^2$$
  - 二次方程式
  - 誤差に対する期待値が f と F で異なる

  $\rightarrow$ 重みを考慮する必要あり。
  $$E = \overline{f^2(\Delta \log f)^2}$$





重みつき最小自乗法

重み $\equiv$ 信頼性

$$E = \overline{w_i \left(f_i - \widetilde{f_i} \right)^2}$$

バラツキが大きいデータは信頼できない。

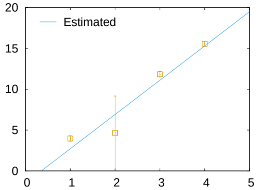
$$\mathrm{E}\left[\left(f_i - \widetilde{f_i} \right)^2 \right] \sim \mathrm{E}\left[\left(f_i - \widehat{f_i} \right)^2 \right] = \sigma_{f_i}^2$$

$$\to \frac{1}{\sigma_{f_i}^2} \mathrm{E}\left[\left(f_i - \widetilde{f_i} \right)^2 \right] \sim 1(\text{constant})$$

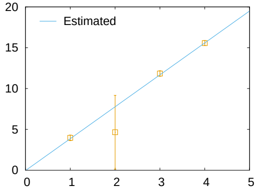
$\to i$ に依存しない $\quad \to \quad \therefore w_i = \dfrac{1}{\sigma_{f_i}^2}$

$$\boxed{E = \overline{\frac{1}{\sigma_{f_i}^2} \left(f_i - \widetilde{f_i} \right)^2} \qquad (1)}$$

- 重みなし



- 重みつき



最小自乗法における変数変換

$$E = \overline{\frac{1}{\sigma_{f_i}^2}\left(f_i - \widetilde{f_i}\right)^2} \qquad (2)$$

$F_i = F(f_i)$ の変換を行う場合,

$$\begin{aligned} f_i - \widetilde{f_i} = \Delta f_i &= \Delta F_i \frac{\Delta f_i}{\Delta F_i} \\ &\simeq (F_i - \widetilde{F_i}) \left.\frac{df}{dF}\right|_i \end{aligned}$$

$$E = \overline{\frac{1}{\left(\frac{dF}{df}\right)_i^2 \sigma_{f_i}^2}\left(F_i - \widetilde{F_i}\right)^2} \qquad (3)$$

$\left|\frac{dF}{df}\right|$ はエラーバーの拡大率に相当

$F = \log f$ の場合

$$\frac{dF}{df} = \frac{1}{f}$$

$$E = \overline{\frac{f_i^2}{\sigma_{f_i}^2}(\log f_i - \log \widetilde{f_i})^2}$$

- $f = \widehat{f} + n$
  n: white $\to \sigma_n^2 = \mathrm{const.}$
- $E = \dfrac{1}{\sigma_n^2}\overline{f_i^2 (\log f_i - \log \widetilde{f_i})^2}$
- 大きな f の重みが大きい