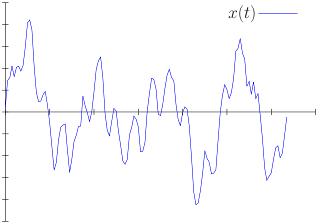
8. スペクトルを用いた雑音の除去
8.1 ウィーナー (Wiener) フィルター

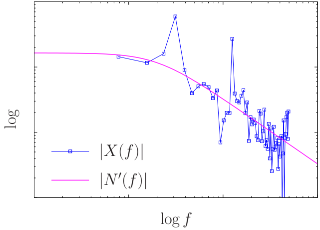
観測モデル：

$$x(t) = \widehat{x}(t) + n(t) \qquad (1)$$

$x(t)$: Observed (known)	$X(f)$ (known)	$\lvert X(f)\rvert$ (known)
$\widehat{x}(t)$: True (Unknown)	$\widehat{X}(f)$ (Unknown)	$\lvert \widehat{X}(f)\rvert$ (Unknown)
$n(t)$: Noise (Unknown)	$N(f)$ (Unknown)	$\lvert N(f)\rvert$ (unknown), $\lvert N'(f)\rvert$ (known)

$\left(\lvert N'(f)\rvert : \lvert N(f)\rvert\right.$ **の代用値**, white/red, $\left.\sigma_{n'}^2, \ldots\right)$





$\downarrow$

$\widehat{x}(t)$?

パーセバルの定理

パーセバルの定理

$$\int_{-\infty}^{\infty} |a(t)|^2 \, dt = \int_{-\infty}^{\infty} |A(f)|^2 \, df \quad (2)$$

$$\begin{cases} A(f) = \int_{-\infty}^{\infty} a(t) e^{-j2\pi f t} \, dt \\ a(t) \;\; = \int_{-\infty}^{\infty} A(f) e^{+j2\pi f t} \, df \end{cases} \quad (3)$$

(証明)

$$\begin{aligned} \text{LHS} &= \int_t \left(\int_f A(f) e^{+j2\pi f t} \, df \cdot \int_{f'} A^*(f') e^{-j2\pi f' t} \, df' \right) dt \\ &= \int_f \int_{f'} A(f) A^*(f') \underbrace{\int_t e^{+j2\pi (f-f')t} \, dt}_{=\delta(f-f')} \, df' \, df \\ &= \int_f \int_{f'} A(f) A^*(f') \, \delta(f - f') \, df' \, df \\ &= \int_f A(f) A^*(f) \, df = \int_{-\infty}^{\infty} |A(f)|^2 \, df = \text{RHS} \end{aligned}$$

無相関信号のスペクトル積

$a(t)$ と $n(t)$ が無相関の場合，

$$\int A^*(f)N(f)\,df = C_{a,n}(0) = 0. \quad (4)$$

無相関信号同士のスペクトルの積は積分すると 0 になる。

証明

$$\int A^*(f)N(f)\,df \qquad (A(f) \text{ と } N(f) \text{ を FT で表現})$$

$$= \int_f \int_t a^*(t)e^{+j2\pi ft}\,dt \int_{t'} n(t')e^{-j2\pi ft'}\,dt'\,df \quad \text{(積分順序の変換)}$$

$$= \int_t \int_{t'} a^*(t)n(t') \int_f e^{+j2\pi f(t-t')}\,df\,dt'\,dt \quad \left(\int e^{+j2\pi f(t-t')}\,df = \delta(t-t')\right)$$

$$= \int_t a^*(t) \int_{t'} n(t')\delta(t-t')\,dt'\,dt = \int_t a^*(t)n(t)\,dt = C_{a,n}(0) = 0.$$

スペクトル空間のフィルター関数 $\Phi_x(f)$

$$x(t) = \widehat{x}(t) + n(t) \quad (x, \widehat{x}, n \in \mathbb{R}) \quad (1)$$

$$X(f) = \widehat{X}(f) + N(f) \quad (X, \widehat{X}, N \in \mathbb{C}) \quad (5)$$

$$\widetilde{X}(f) = X(f)\Phi_x(f) \quad (6)$$

$$\widetilde{x}(t) = \mathcal{F}^{-1}\left\{\widetilde{X}(f)\right\} \quad (7)$$

$\Phi_x(f) \in \mathbb{R}$: フィルター関数
$\widetilde{X}(f) \in \mathbb{C}$: 推定スペクトル
$\widetilde{x}(t)$: 推定信号

- 推定値 $\widetilde{x}(t)$ を残差が最小となる解を最小自乗法により決定する。

  (残差積分)

$$E = \int |\widetilde{x}(t) - \widehat{x}(t)|^2 \, dt \quad (8)$$

- パーセバルの定理より

$$E = \int |\widetilde{x}(t) - \widehat{x}(t)|^2 \, dt = \int \underbrace{\left|\widetilde{X}(f) - \widehat{X}(f)\right|^2}_{\equiv I(f) \geq 0} df \quad (9)$$

- 積分核 $I(f) \geq 0$ なので，

  Minimize $E \Leftrightarrow$ Minimize $I(f)$ for all f.

$$\rightarrow \quad \frac{\partial E}{\partial \Phi_x} = 0 \quad (10)$$

  (停留 (Stationary) 条件)

- $$E = \int \left|\widetilde{X} - \widehat{X}\right|^2 df = \int \left|X\Phi_x - \widehat{X}\right|^2 df$$

  E は関数 Φ_x の関数

  関数の関数を，汎関数と呼ぶ

$$I = \left|\widetilde{X} - \widehat{X}\right|^2 = \left|X\Phi_x - \widehat{X}\right|^2$$
$$= \left|(\widehat{X} + N)\Phi_x - \widehat{X}\right|^2 = \left|\widehat{X}(\Phi_x - 1) + N\Phi_x\right|^2$$
$$= (\widehat{X}(\Phi_x - 1) + N\Phi_x)^*(\widehat{X}(\Phi_x - 1) + N\Phi_x)$$
$$= \underbrace{\left|\widehat{X}\right|^2 (\Phi_x - 1)^2 + |N|^2 \Phi_x^2}_{\equiv I'}$$
$$+ \underbrace{(\widehat{X}^* N + \widehat{X} N^*)(\Phi_x - 1)\Phi_x}_{f \text{ に関する積分は } 0(\because \text{ Eq.}(4))}$$

$$E = \int I\, df = \int I'\, df$$
$$I' = \left|\widehat{X}\right|^2 (\Phi_x - 1)^2 + |N|^2 \Phi_x^2 \geq 0$$

minimize E $\Leftrightarrow$ minimize I' $\Leftrightarrow$ $\dfrac{\partial I'}{\partial \Phi_x} = 0$

$$\frac{\partial I'}{\partial \Phi_x} = 2\left(\left|\widehat{X}\right|^2 (\Phi_x - 1) + |N|^2 \Phi_x\right) = 0$$
$$\rightarrow \Phi_x = \frac{\left|\widehat{X}\right|^2}{\left|\widehat{X}\right|^2 + |N|^2}$$

Wiener filter

$$\Phi_x(f) = \frac{\left|\widehat{X}(f)\right|^2}{\left|\widehat{X}(f)\right|^2 + |N(f)|^2} \quad (11)$$
$$(0 \leq \Phi_x(f) \leq 1)$$

ただし，この形式では，
真の解の FT ($\widehat{X}(f)$) が必要

観測値によるフィルター関数の表現

$$\begin{pmatrix} X = \widehat{X} + N \\ \widetilde{X} = X\Phi_x \end{pmatrix}$$

$$I = \left|\widetilde{X} - \widehat{X}\right|^2 = |X\Phi_x - (X - N)|^2$$
$$= |X(\Phi_x - 1) + N|^2$$
$$= (X(\Phi_x - 1) + N)^*(X(\Phi_x - 1) + N)$$
$$= |X|^2 (\Phi_x - 1)^2 + |N|^2 + (X^*N + XN^*)(\Phi_x - 1)$$

$$\begin{pmatrix} X^*N + XN^* \\ = (\widehat{X} + N)^*N + (\widehat{X} + N)N^* \\ = \widehat{X}^*N + \widehat{X}N^* + 2|N|^2 \end{pmatrix}$$

$$= |X|^2 (\Phi_x - 1)^2 - |N|^2 + 2|N|^2 \Phi_x + (\widehat{X}^*N + \widehat{X}N^*)(\Phi_x - 1)$$

$\int A^*N\,df = 0$ (Eq.(4)) より,
最後の項は考慮しなくても E は同じ。

$$E = \int I\,df = \int I'\,df$$

$$I' = |X|^2 (\Phi_x - 1)^2 - |N|^2 + 2|N|^2 \Phi_x$$

$$\frac{\partial I'}{\partial \Phi_x} = 2\left(|X|^2 (\Phi_x - 1) + |N|^2\right) = 0$$

$$\rightarrow \quad \Phi_x = \frac{|X|^2 - |N|^2}{|X|^2} \simeq \frac{|X|^2 - |N'|^2}{|X|^2}$$

$(\because N' \simeq N)$

(右辺は全て既知)

$\therefore$

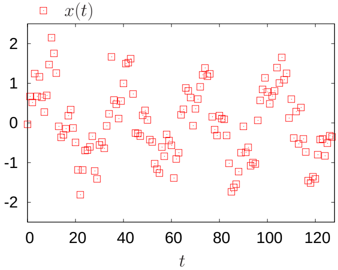
$$\Phi_x(f) = \frac{|X(f)|^2 - |N'(f)|^2}{|X(f)|^2} \quad (12)$$

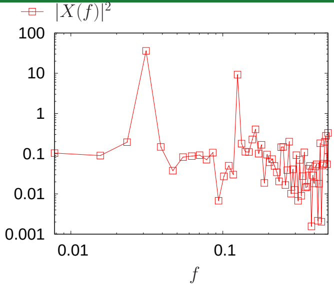
Wiener filter の適用の手順

1. $|X(f)|^2$
   1. $x(t)$ の測定
   2. $|X(f)|^2 = |\mathcal{F}\{x(t)\}|^2$
2. $|N'(f)|^2$
   - $n'(t)$ の測定
     (測定対象なし)
     $|N'(f)|^2 = |\mathcal{F}\{n'(t)\}|^2$
   - 不可能な場合には
     ノイズの性質を考慮して決める
     - White : $|N'(f)| = \mathrm{const}$
     - Brownian : $|N'(f)| \propto \frac{1}{f^2+a^2}$
3. $\Phi_x(f) = \frac{|X(f)|^2 - |N'(f)|^2}{|X(f)|^2}$
   - If $\Phi_x(f) < 0$, $\Phi_x(f) = 0$.
     (非可逆)

   Wiener フィルターは，振幅のみを変更し，位相は変えない。
   $(\Phi_x < 0 \Leftrightarrow |\Phi_x| e^{-i\pi})$
4. $\tilde{x}(t)$
   1. $\tilde{X}(f) = X(f)\Phi_x(f)$
   2. $\tilde{x}(t) = \mathcal{F}^{-1}\left\{\tilde{X}(f)\right\}$

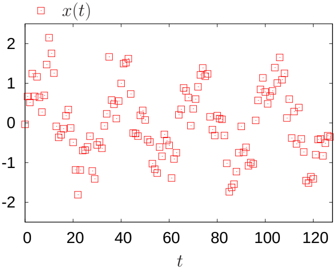
Wiener filter の適用例

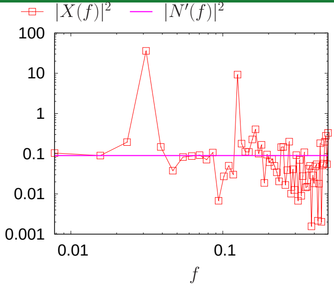




1 $x(t)$, $X(f)$

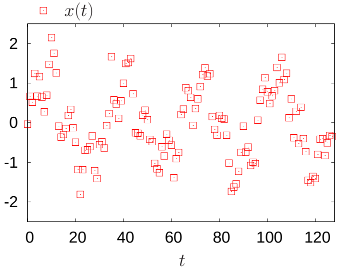
Wiener filter の適用例

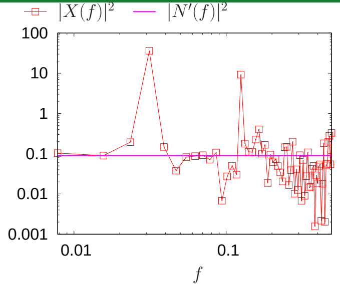




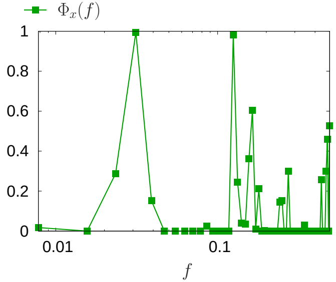
1. $x(t)$, $X(f)$
2. $N'(f)$

Wiener filter の適用例

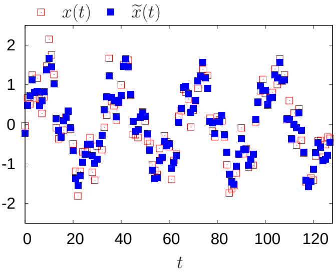


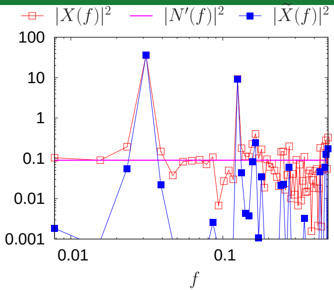


1. $x(t)$, $X(f)$
2. $N'(f)$
3. $\Phi_x(f) = \frac{|X(f)|^2 - |N'(f)|^2}{|X(f)|^2}$

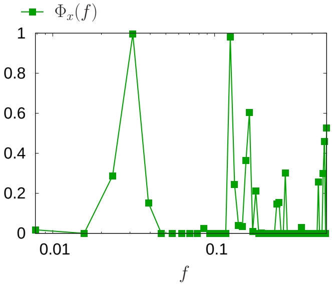


Wiener filter の適用例





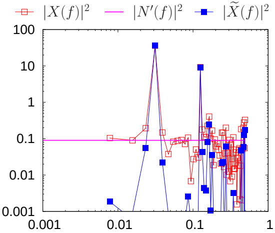
1. $x(t)$, $X(f)$
2. $N'(f)$
3. $\Phi_x(f) = \frac{|X(f)|^2 - |N'(f)|^2}{|X(f)|^2}$
4. $\widetilde{X}(f)$, $\widetilde{x}(t)$

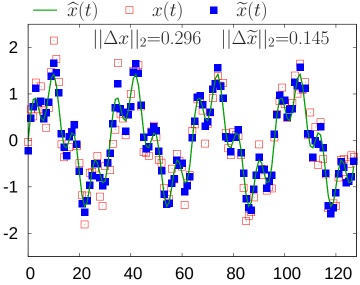


$|N'(f)|$ の効果 (白色雑音)

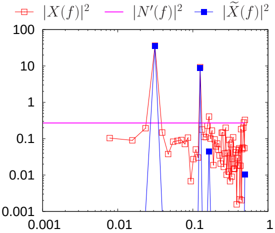
$$\left(x(t) = \sin\left(\frac{2\pi t}{T/4}\right) + \frac{1}{2}\sin\left(\frac{2\pi t}{T/16}\right) + n, \quad \sigma_n^2 = 0.09\right)$$

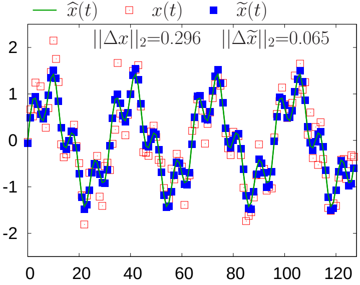
$|N'| = \sigma_n^2$
$= 0.09$





$|N'| = 3\sigma_n^2$
$= 0.27$



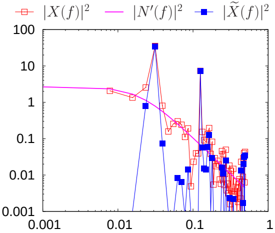


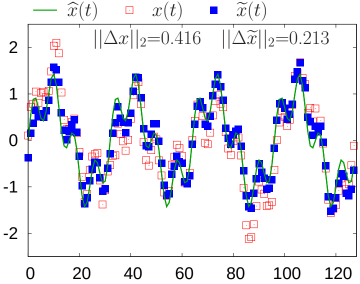
$|N'(f)|$ の効果 (ブラウニアン雑音)

$$\left(x(t) = \sin\left(\frac{2\pi t}{T/4}\right) + \frac{1}{2}\sin\left(\frac{2\pi t}{T/16}\right) + r, \quad \alpha = 0.02, \quad \sigma_{n'}^2 = 0.04 \quad (|N'(0)|^2 = 2.7)\right)$$

$\alpha = 0.02,$

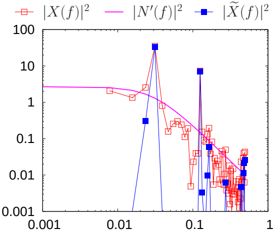
$|N'(0)|^2$
$= 2.7$

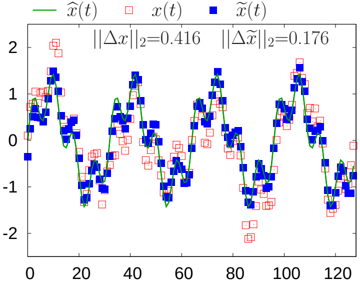




$\alpha = 0.03,$

$|N'(0)|^2$
$= 2.7$

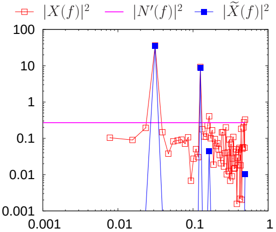


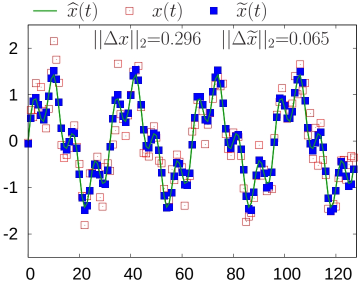


Wiener filter と移動平均の比較

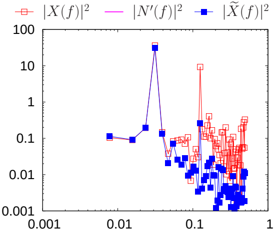
$$\left(x(t) = \sin\left(\frac{2\pi t}{T/4}\right) + \frac{1}{2}\sin\left(\frac{2\pi t}{T/16}\right) + n, \quad \sigma_n^2 = 0.09\right)$$

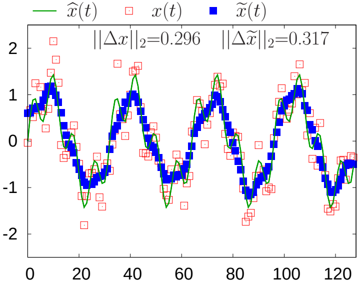
Wiener filter
$(|N'|^2 = 0.27)$





Mov. Ave
$(N_m = 7)$





Wiener filter のまとめ

$x(t)$ と $|N'(f)|$ が既知のとき，

$$X(f) = \mathcal{F}\{x(t)\}$$

$$\Phi_x(f) = \frac{|X(f)|^2 - |N'(f)|^2}{|X(f)|^2}$$

$$\widetilde{X}(f) = X(f)\Phi_x(f)$$

$$\widetilde{x}(t) = \mathcal{F}^{-1}\left\{\widetilde{X}(f)\right\}$$

- 雑音のスペクトルを考慮することができる。
- 移動平均と異なり，信号のスペクトルに複数のピークがあっても適用できる。

Wiener filter は最適化フィルターとも呼ばれる。