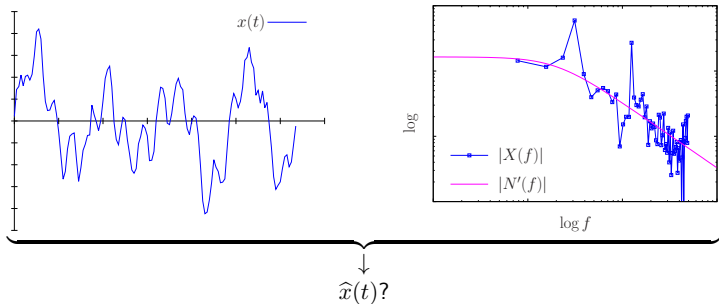


8. スペクトルを用いた雑音の除去

8.1 ウィーナー (Wiener) フィルター

観測モデル : $x(t) = \hat{x}(t) + n(t)$ (1)

$x(t)$: Observed (known)	$X(f)$ (known)	$ X(f) $ (known)
$\hat{x}(t)$: True (Unknown)	$\hat{X}(f)$ (Unknown)	$ \hat{X}(f) $ (Unknown)
$n(t)$: Noise (Unknown)	$N(f)$ (Unknown)	$ N(f) $ (unknown), $ N'(f) $ (known)
$\left(N'(f) : N(f) \text{ の代用値, } \right.$ $\left. \text{white/red, } \sigma_n^2, \dots \right)$		



パーセバルの定理

パーセバルの定理

$$\int_{-\infty}^{\infty} |a(t)|^2 dt = \int_{-\infty}^{\infty} |A(f)|^2 df \quad (2)$$

$$\begin{cases} A(f) = \int_{-\infty}^{\infty} a(t) e^{-j2\pi ft} dt \\ a(t) = \int_{-\infty}^{\infty} A(f) e^{+j2\pi ft} df \end{cases} \quad (3)$$

(証明)

$$\begin{aligned} \text{LHS} &= \int_t \left(\int_f A(f) e^{+j2\pi ft} df \cdot \int_{f'} A^*(f') e^{-j2\pi f't} df' \right) dt \\ &= \int_f \int_{f'} A(f) A^*(f') \underbrace{\int_t e^{+j2\pi(f-f')t} dt}_{=\delta(f-f')} df' df \\ &= \int_f \int_{f'} A(f) A^*(f') \delta(f - f') df' df \\ &= \int_f A(f) A^*(f) df = \int_{-\infty}^{\infty} |A(f)|^2 df = \text{RHS} \end{aligned}$$

無相関信号のスペクトル積

$a(t)$ と $n(t)$ が無相関の場合,

$$\int A^*(f)N(f) df = C_{a,n}(0) = 0. \quad (4)$$

無相関信号同士のスペクトルの積は積分すると 0 になる。

証明

$$\begin{aligned} & \int A^*(f)N(f) df \quad (A(f) \text{ と } N(f) \text{ を FT で表現}) \\ &= \int_f \int_t a^*(t)e^{+j2\pi ft} dt \int_{t'} n(t')e^{-j2\pi ft'} dt' df \quad (\text{積分順序の変換}) \\ &= \int_t \int_{t'} a^*(t)n(t') \int_f e^{+j2\pi f(t-t')} df dt' dt \quad \left(\int e^{+j2\pi f(t-t')} df = \delta(t-t') \right) \\ &= \int_t a^*(t) \int_{t'} n(t')\delta(t-t') dt' dt = \int_t a^*(t)n(t) dt = C_{a,n}(0) = 0. \end{aligned}$$

スペクトル空間のフィルター関数 $\Phi_x(f)$

$$x(t) = \hat{x}(t) + n(t) \quad (x, \hat{x}, n \in \mathbb{R}) \quad (1)$$

$$X(f) = \hat{X}(f) + N(f) \quad (X, \hat{X}, N \in \mathbb{C}) \quad (5)$$

$$\tilde{X}(f) = X(f)\Phi_x(f) \quad (6)$$

$$\tilde{x}(t) = \mathcal{F}^{-1} \left\{ \tilde{X}(f) \right\} \quad (7)$$

$\Phi_x(f) \in \mathbb{R}$: フィルター関数

$\tilde{X}(f) \in \mathbb{C}$: 推定スペクトル

$\tilde{x}(t)$: 推定信号

- 推定値 $\tilde{x}(t)$ を残差が最小となる解を最小自乗法により決定する。

(残差積分)

$$E = \int |\tilde{x}(t) - \hat{x}(t)|^2 dt \quad (8)$$

- パーセバルの定理より

$$\begin{aligned} E &= \int |\tilde{x}(t) - \hat{x}(t)|^2 dt \\ &= \int \underbrace{|\tilde{X}(f) - \hat{X}(f)|^2}_{\equiv I(f) \geq 0} df \quad (9) \end{aligned}$$

- 積分核 $I(f) \geq 0$ なので,

$$\begin{aligned} \text{Minimize } E &\Leftrightarrow \text{Minimize } I(f) \\ &\text{for all } f. \\ \rightarrow \frac{\partial E}{\partial \Phi_x} &= 0 \quad (10) \end{aligned}$$

(停留 (Stationary) 条件)

-

$$E = \int |\tilde{X} - \hat{X}|^2 df = \int |X\Phi_x - \hat{X}|^2 df$$

E は関数 Φ_x の関数

関数の関数を, 汎関数と呼ぶ

$$\begin{aligned}
 I &= \left| \tilde{X} - \hat{X} \right|^2 = \left| X\Phi_x - \hat{X} \right|^2 \\
 &= \left| (\hat{X} + N)\Phi_x - \hat{X} \right|^2 = \left| \hat{X}(\Phi_x - 1) + N\Phi_x \right|^2 \\
 &= (\hat{X}(\Phi_x - 1) + N\Phi_x)^* (\hat{X}(\Phi_x - 1) + N\Phi_x) \\
 &= \underbrace{\left| \hat{X} \right|^2 (\Phi_x - 1)^2 + |N|^2 \Phi_x^2}_{\equiv I'} \\
 &\quad + \underbrace{(\hat{X}^* N + \hat{X} N^*) (\Phi_x - 1) \Phi_x}_{f \text{ に関する積分は } 0(\cdot \text{ Eq.(4)})}
 \end{aligned}$$

$$E = \int I df = \int I' df$$

$$I' = \left| \hat{X} \right|^2 (\Phi_x - 1)^2 + |N|^2 \Phi_x^2 \geq 0$$

$$\text{minimize } E \Leftrightarrow \text{minimize } I' \Leftrightarrow \frac{\partial I'}{\partial \Phi_x} = 0$$

$$\begin{aligned}
 \frac{\partial I'}{\partial \Phi_x} &= 2 \left(\left| \hat{X} \right|^2 (\Phi_x - 1) + |N|^2 \Phi_x \right) = 0 \\
 \rightarrow \Phi_x &= \frac{|\hat{X}|^2}{|\hat{X}|^2 + |N|^2}
 \end{aligned}$$

Wiener filter

$$\begin{aligned}
 \Phi_x(f) &= \frac{|\hat{X}(f)|^2}{|\hat{X}(f)|^2 + |N(f)|^2} \\
 (0 \leq \Phi_x(f) \leq 1)
 \end{aligned} \tag{11}$$

ただし、この形式では、
真の解の FT ($\hat{X}(f)$) が必要

観測値によるフィルター関数の表現

$$\begin{aligned}
 & \begin{pmatrix} X = \hat{X} + N \\ \tilde{X} = X\Phi_x \end{pmatrix} \\
 I &= \left| \tilde{X} - \hat{X} \right|^2 = |X\Phi_x - (X - N)|^2 \\
 &= |X(\Phi_x - 1) + N|^2 \\
 &= (X(\Phi_x - 1) + N)^*(X(\Phi_x - 1) + N) \\
 &= |X|^2 (\Phi_x - 1)^2 + |N|^2 \\
 &\quad + (X^*N + XN^*)(\Phi_x - 1) \\
 & \begin{pmatrix} X^*N + XN^* \\ = (\hat{X} + N)^*N + (\hat{X} + N)N^* \\ = \hat{X}^*N + \hat{X}N^* + 2|N|^2 \end{pmatrix} \\
 &= |X|^2 (\Phi_x - 1)^2 - |N|^2 + 2|N|^2 \Phi_x \\
 &\quad + (\hat{X}^*N + \hat{X}N^*)(\Phi_x - 1)
 \end{aligned}$$

$\int A^*N df = 0$ (Eq.(4)) より,
最後の項は考慮しなくても E は同じ。

$$\begin{aligned}
 E &= \int I df = \int I' df \\
 I' &= |X|^2 (\Phi_x - 1)^2 - |N|^2 + 2|N|^2 \Phi_x \\
 \frac{\partial I'}{\partial \Phi_x} &= 2 \left(|X|^2 (\Phi_x - 1) + |N|^2 \right) = 0 \\
 \rightarrow \Phi_x &= \frac{|X|^2 - |N|^2}{|X|^2} \simeq \frac{|X|^2 - |N'|^2}{|X|^2} \\
 &(\because N' \simeq N)
 \end{aligned}$$

(右辺は全て既知)

$\therefore$

$$\Phi_x(f) = \frac{|X(f)|^2 - |N'(f)|^2}{|X(f)|^2} \quad (12)$$

Wiener filter の適用の手順

1 $|X(f)|^2$

① $x(t)$ の測定

② $|X(f)|^2 = |\mathcal{F}\{x(t)\}|^2$

2 $|N'(f)|^2$

▶ $n'(t)$ の測定
(測定対象なし)

$$|N'(f)|^2 = |\mathcal{F}\{n'(t)\}|^2$$

▶ 不可能な場合には
ノイズの性質を考慮して決める

- White : $|N'(f)| = \text{const}$
- Brownian : $|N'(f)| \propto \frac{1}{f^2 + a^2}$

3 $\Phi_x(f) = \frac{|X(f)|^2 - |N'(f)|^2}{|X(f)|^2}$

▶ If $\Phi_x(f) < 0$, $\Phi_x(f) = 0$.
(非可逆)

Wiener フィルターは、振幅のみを変更し、位相は変えない。

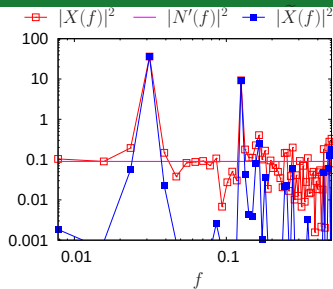
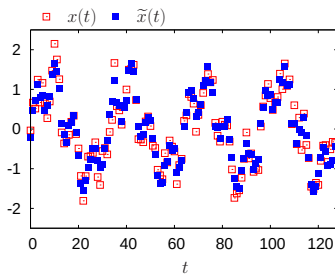
$$(\Phi_x < 0 \Leftrightarrow |\Phi_x|e^{-i\pi})$$

4 $\tilde{x}(t)$

① $\tilde{X}(f) = X(f)\Phi_x(f)$

② $\tilde{x}(t) = \mathcal{F}^{-1}\{\tilde{X}(f)\}$

Wiener filter の適用例

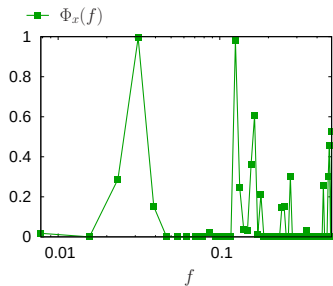


1 $x(t), X(f)$

2 $N'(f)$

3 $\Phi_x(f) = \frac{|X(f)|^2 - |N'(f)|^2}{|X(f)|^2}$

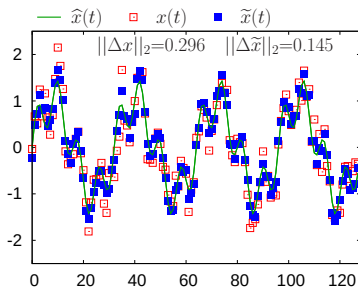
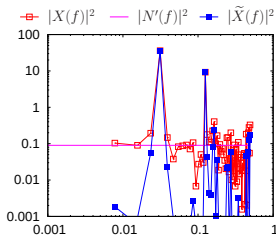
4 $\tilde{X}(f), \tilde{x}(t)$



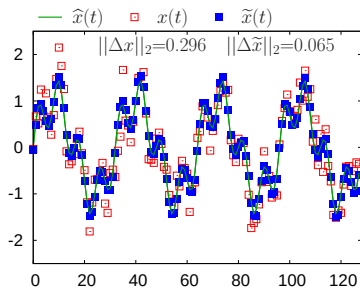
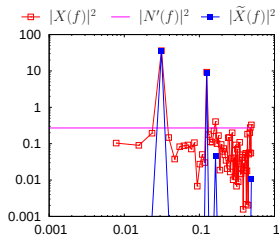
$|N'(f)|$ の効果 (白色雑音)

$$x(t) = \sin\left(\frac{2\pi t}{T/4}\right) + \frac{1}{2} \sin\left(\frac{2\pi t}{T/16}\right) + n, \quad \sigma_n^2 = 0.09$$

$$|N'| = \sigma_n^2 = 0.09$$

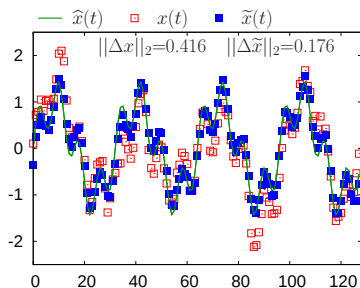
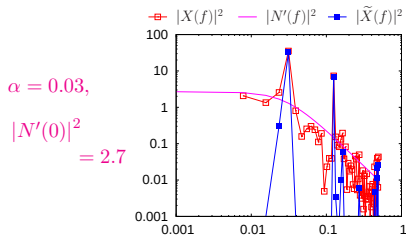
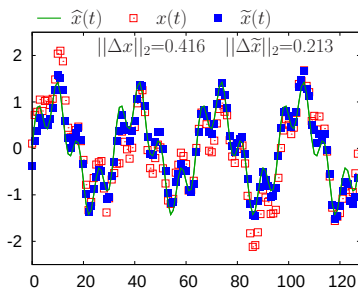
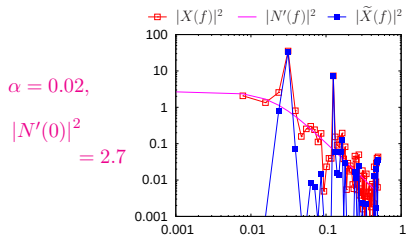


$$|N'| = 3\sigma_n^2 = 0.27$$



$|N'(f)|$ の効果 (ブラウン雑音)

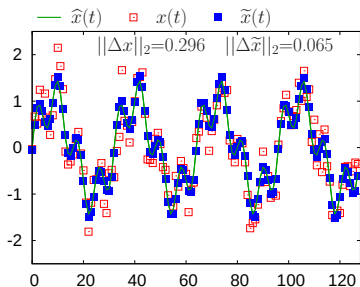
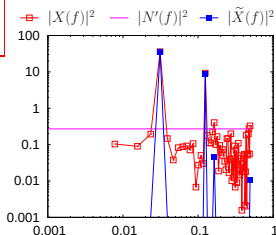
$$\left(x(t) = \sin\left(\frac{2\pi t}{T/4}\right) + \frac{1}{2} \sin\left(\frac{2\pi t}{T/16}\right) + r, \quad \alpha = 0.02, \quad \sigma_n^2 = 0.04 \quad (|N'(0)|^2 = 2.7) \right)$$



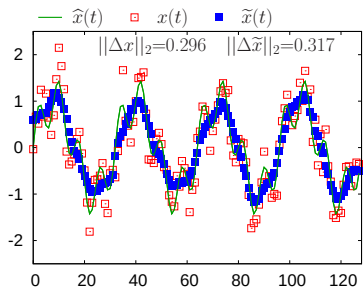
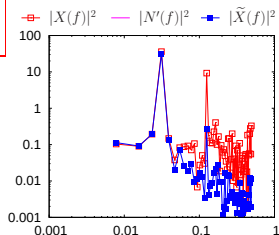
Wiener filter と移動平均の比較

$$x(t) = \sin\left(\frac{2\pi t}{T/4}\right) + \frac{1}{2} \sin\left(\frac{2\pi t}{T/16}\right) + n, \quad \sigma_n^2 = 0.09$$

Wiener filter
($|N'|^2 = 0.27$)



Mov. Ave
($N_m = 7$)



Wiener filter のまとめ

$x(t)$ と $|N'(f)|$ が既知のとき,

$$X(f) = \mathcal{F}\{x(t)\}$$

$$\Phi_x(f) = \frac{|X(f)|^2 - |N'(f)|^2}{|X(f)|^2}$$

$$\tilde{X}(f) = X(f)\Phi_x(f)$$

$$\tilde{x}(t) = \mathcal{F}^{-1}\{\tilde{X}(f)\}$$

- 雑音のスペクトルを考慮することができる。
- 移動平均と異なり、信号のスペクトルに複数のピークがあっても適用できる。

Wiener filter は最適化フィルターとも呼ばれる。