Deconvolution of blurred images without noise

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According to the convolution theorem, a convolution integral of two signals is expressed as a product of those Fourier transforms as follows.

$$o(x, y) = \iint g(x', y')\, i(x - x', y - y')\, dx'\, dy', \tag{1}$$

$$O(k_x, k_y) = G(k_x, k_y) \cdot I(k_x, k_y), \tag{2}$$

where the capital letters denote the Fourier transformed functions.

When images of a pair input i_{ref} and output o_{ref} are known, a response function $G(k_x, k_y)$ can be determined by the following equation, even when $g(x, y)$ was not measured.

$$G = \frac{O_{\text{ref}}}{I_{\text{ref}}}. \tag{3}$$

When the denominator on the right-hand side approaches to zero, G diverges and numerical error increases. To suppress the divergence, a numerical treatment is required such as adding a small amount of artificial noise to the denominator. If the denominator is a complex-valued function, the artificial positive real-valued noise is added after the following rationalization of the complex-valued function:

$$G = \frac{O_{\text{ref}} I^*_{\text{ref}}}{|I_{\text{ref}}|^2} \simeq \frac{O_{\text{ref}} I^*_{\text{ref}}}{|I_{\text{ref}}|^2 + \epsilon_I}, \tag{4}$$

where the superscript '*' denotes a complex conjugate. The quantity ϵ_I is chosen from $\overline{|I_{\text{ref}}|}$ which is an average of $|I_{\text{ref}}|^2$,

$$\epsilon_I = \alpha \overline{|I_{\text{ref}}|^2}. \tag{5}$$

To avoid the numerical rounding error, $\alpha > 10^{-15}$ is suitable for the case where the numerical precision of real-values is double precision.

Once G is obtained, $\tilde{I}_{\text{test}}$, which is a Fourier transformed function of a test input function, can be estimated from the other measurement taken by the same system as

$$\tilde{I}_{\text{test}} = \frac{O_{\text{test}}}{G} \simeq \frac{O_{\text{test}} G^*}{|G|^2 + \epsilon_G} = F O_{\text{test}}, \tag{6}$$

$$F \equiv \frac{G^*}{|G|^2 + \epsilon_G}, \tag{7}$$

where F is a filter to obtain $\tilde{I}_{\text{test}}$ from O_{test} and ϵ_G can be determined by the same way of ϵ_I. When $|G|$ is small, F becomes large. In some cases, only a use of ϵ_G can not suppress the divergence sufficiently. In such cases, the coupling of the limiting of F is effective.

$$F' = \begin{cases} F & (|G| \geq G_{\text{lim}}), \\ 0 & (|G| < G_{\text{lim}}). \end{cases} \tag{8}$$

By replacing F with F', $\tilde{I}$ dose not diverge, and by applying inverse Fourier transform $\tilde{i}(x, y)$ can be evaluated. When $G_{\text{lim}} \sim 10^{-8}$, we can recognize a character pattern from the deconvoluted result $\tilde{i}$.

In the above procedure, there are two times of divisions. Although we can reduce the number of times of divisions to one, it is preferable to monitor the functions for each step to understand the propagation of information.

The above procedures are summarized as follows.

1. Evaluation of I_{ref}, O_{ref}, O_{test} which are Fourier transform of i_{ref}, o_{ref}, o_{test}, respectively.

2. Evaluation of G from I_{ref} and O_{ref}. To suppress the divergence of G, a modification of the denominator is required in general. If you wish, a spread function g can be also evaluated by applying inverse Fourier transform. (In the generation of o, a Gaussian function with width of 10 pixels was used for g.)

3. Evaluation of F with limiting of $|G|$. Not only the numerical treatment to suppress the divergence of F similar to G but also setting an upper limit of F is required to improve the quality of the estimation.

4. Evaluation of $\tilde{I}_{\text{test}}$ by taking a product of O_{test} and F.

5. Evaluation of $\tilde{i}_{\text{test}}$ by using inverse Fourier transform.

Monitoring each evaluated function step by step is an easy way to debug.